Pricing and hedging of a general kind of multiasset option*

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1 Introduction

In this paper we present a closed formula for the price and for the replicating strategy of a European option on several assets having the general payoff

$$C_T = \sum_{j=1}^{N} (\langle \phi_j, \bar{S}(T)\rangle - K_j)\mathbf{1}_{\mathcal{E}_j} \ ,$$

where ϕ_j and K_j, $j = 1, \ldots, N$, are respectively n-dimensional real vectors and constants, $\bar{S}$ is a vector of n assets, each one traded in a (possibly) different country and converted in the domestic currency via the exchange rate $X_{1j}(t)$ at time t, that is

$$\bar{S}(t) = (\bar{S}_1(t), \ldots, \bar{S}_n(t)), \quad \text{where} \quad \bar{S}_j(t) = X_{1j}(t)S_j(t), \quad t \in [0,T], \quad j = 2,3,\ldots,n$$

with the understanding that $S_1(\cdot) = B_1(\cdot, T)$ is the zero coupon bond with maturity T traded in the domestic country, for which the exchange rate is $X_{11} \equiv 1$, S_j is traded in the j-th country, for $j = 2, \ldots, n$, $\langle \cdot, \cdot \rangle\rangle$ is the euclidean scalar product in $\mathbb{R}^n$, and the $\mathcal{E}_j$ are disjoint sets in Ω depending on the value of $\bar{S}(T)$. In our model, besides the assumption of absence of arbitrage opportunities, we make the further hypotheses that the quantities significant to our analysis (that is the prices $S_j(t)$ in t of the j-th asset, the exchange rates $X_{ij}(t)$ between the i-th country and the j-th one, and the spot forward rate $r_j(t,x)$ in time t with maturity $t+x$ of the j-th country) satisfy stochastic differential equations driven by a k-dimensional Wiener process. This leads us to a specific structure for the stochastic differential equations satisfied by these quantities. If we make the further assumption that the market is complete and, if we consider the quantities cited above as a whole process, it is Markov, then we are able to linearize the price of the multiple option and to get a formula in terms of a linear combination of the assets S_j, weighted by the probabilities of suitable exercise sets. In order to arrive to such a formula, we change the numeraire in each of the elements of the linear combination using for each country j the corresponding risk-neutral probability $\mathbb{Q}_j^S$, and the forward-neutral probability $\mathbb{Q}_1^T$, introduced in El Karoui *et al.* (1995) and Jamshidian (1989). With the additional assumptions that the risk premium and the diffusion term of the forward rates are deterministic in all the countries, we are

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able to derive explicit formulae both for the price and for the hedging portfolio of the multiple option. Finally we present a case in which our formula is reduced to the well known Johnson's formula for the option on the maximum on several assets (see Johnson,1987), and two applications of our option, namely the MAP strategy (Multiple Asset Performance), presented in Fong and Vasicek (1989), and the option on the arithmetic mean of $n-1$ assets.

This work generalises results contained in Romagnoli and Vargiolu (1998), and uses three different topics in finance: international finance, term structure of interest rates and options on several assets, focusing much on the third topic. The first relevant work in international finance is Garman and Kohlagen (1983), which is the seminal paper in the subject in the same sense as the paper of Black and Scholes (1973) is in pricing and hedging of European options. The main idea presented in the Garman-Kohlagen model is the assumption of absence of arbitrage opportunities between the countries, formalized by the fact that the foreign prices expressed in terms of the domestic currency by the exchange rate behaves as a domestic price. Through this approach we may see the exchange rate as a domestic asset that pays a continous dividend which corresponds to the foreign interest rate. Term structure of interest rates is a wide topic in finance, but here we do not concentrate much on it. We only notice that, though in a first approximation it is possible to suppose that the rates are deterministic, since we are dealing with exchange rates, we suppose that the interest rates in our n countries are stochastic. In order to justify this approach, we notice that our main application is the MAP strategy, that is a derivative asset on stock indexes of different countries, so the exchange risk is comparable to the risk of the considered assets. In order to derive explicit formulae, though, we suppose that the volatility of the bonds and the risk premiums in all the n countries are deterministic. A deterministic volatility of the bonds is equivalent to the hypothesis that the forward rates are gaussian processes (see for example Amin and Jarrow, 1991, Heath *et al.,* 1992, and Musiela,1993). In particular, we use the Musiela parametrisation (see Brace and Musiela, 1994, Musiela, 1993, or Vargiolu, 1999) for the interest rates rather than the Heath-Jarrow-Morton one (see Heath *et al.,* 1992), because the first easily allows to see that the instantaneous forward rates $(r_i(t,\cdot))_t$ of the different countries $i=1,\ldots,n$ are Markov processes. Coming to the third topic involved in this work, the first work about options on several assets is the paper by Margrabe (1978), that analyses the simplest case of an option to exchange one asset for another. Later, Stulz (1982) arrived to price an option on the maximum of two assets. His work was generalized by Johnson (1987), who solved the same problem for a general number of assets. The first and the last work follow more or less the same idea, that is to linearize the payoff function and to find closed formulas in terms of a ponderate sum of different probabilities calculated in different exercise sets in terms of the multivariate gaussian distribution function. In particular we follow Johnson's technique and find an expression similar to Johnson's formula. This also allows us to find a replicating strategy in the n assets: this derives intuitively from Johnson's formula, even if Johnson himself does not derive the replication strategy from it. Moreover in this work there is the further complication of stochastic interest rates. Because of the gaussian model we have chosen, though, this complication affects the general structure of the price and replication formulae only in terms of more complicated coefficients than those of Johnson.

The paper is organized as follows: in Section 2 we present the model we used; in Section 3 we derive the pricing formula for the multiple option; in Section 4 we derive the hedging strategy from the pricing formula we have found; in Section 5 we show that under stronger assumptions our formula is reduced to Johnson's formula (Johnson, 1987); in Section 6 we present an example of application of our option, namely Fong-Vasicek's MAP strategy (Fong and Vasicek, 1989); in Section 7 we present another example of application, namely the option on the arithmetic mean of several assets.

2 The model

We consider n assets, each one traded in a country, that could be all different ones, but this is not necessary; anyway we assume that asset 1 is always traded in our domestic country, while assets $2, \ldots, n$ could be traded in the domestic country or in a number up to $n-1$ of foreign ones. ¿From now on, we will indicate as "the i-th country" or "the i-th currency" respectively the country or the currency relative to the i-th asset, so if the i-th and the j-th assets are traded in the same country, the i-th and the j-th country will coincide, as their currency will. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we represent the information at time $t \in [0, T]$ with a filtration $(\mathcal{F}_t)_t$, which for technical purposes we assume to be complete and right continuous. We also suppose that all the processes are adapted to the filtration $(\mathcal{F}_t)_t$. We name:

$r_i(t, \theta)$ the instantaneous forward rate prevailing at time t for the maturity $t + \theta = T$ in the i-th country; we will indicate shortly the spot rate $r_i(t, 0)$ with $r_i(t)$;

$S_0(t)$ the price at time t of the riskless asset in the domestic country, expressed in the domestic currency

$S_1(t) = B_1(t, T)$ the price at time t of the zero coupon bond with maturity T in the domestic country

$S_i(t)$ the price at time t of an asset in the i-th country, expressed in the i-th currency

$X_{ij}(t)$ the exchange rate from the i-th country to the j-th country, that is the price of 1 unity of the j-th currency expressed in i-th currency; if the i-th and the j-th currency coincide, then $X_{ij} \equiv 1$.

We suppose that the dynamic of the processes under the historic probability $\mathbb{P}$ are

$$\begin{array}{lcl} dS_0(t) & = & S_0(t) r_1(t)\ dt \\ dB_1(t, T) & = & B_1(t, T)(\mu_1(t, T)\ dt + \langle \Gamma_1(t, T), d\hat{W}(t) \rangle) \\ dS_i(t) & = & S_i(t)(\mu_i(t)\ dt + \langle \sigma_i(t), d\hat{W}(t) \rangle) \\ dr_i(t, \theta) & = & \alpha_i(t, \theta)\ dt + \langle \tau_i(t, \theta), d\hat{W}(t) \rangle \\ dX_{ij}(t) & = & X_{ij}(t)(m(t)\ dt + \langle \sigma_{ij}^X(t), d\hat{W}(t) \rangle) \end{array} \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the euclidean scalar product on $\mathbb{R}^k$, and $(\hat{W}(t))_t$ is a k-dimensional brownian motion which represent the sources of risk which affect the different economies, and where we suppose that all the quantities we introduced satisfy the technical regularity conditions for the integrals to be defined.

We suppose there are no arbitrage opportunities in the domestic market; we shall see in the next theorem that this implies the existence of a probability measure $\mathbb{Q}_1$ equivalent to $\mathbb{P}$, under which the actualized prices of all the domestic assets are martingales, and that there are several contraints on the quantities introduced before.

Lemma 1 If there are no arbitrage opportunities, then it exists a process λ_1 with values in $\mathbb{R}^n$, called *risk premium*, such that

$$\mu_1(t) = r_1(t)\underline{1} + \sigma_1(t)\lambda_1(t)$$

where $\underline{1} = (1, \dots, 1)$.

Proof. If $\sigma_1(t)$ is surjective, then the thesis is obvious; if $\sigma_1(t)$ is not surjective, we build a self-financing strategy H such that $(H_i(t)S_i(t))_i$ is the kernel of $\sigma_1^*(t)$, then the portfolio V defined by H satisfies:

$$dV(t) = r_1(t)V(t)\ dt + \langle (H_i(t)S_i(t))_i\, , \mu_1(t) - r_1(t)\underline{1}\rangle\ dt$$

We notice that this portfolio is defined by a bounded variation process; this implies, by the absence of arbitrage opportunities, that its drift is $r_1(t)V(t)$, so that $\langle (H_i(t), S_i(t))_i\, , \mu_1(t) - r_1(t)\underline{1}\rangle = 0$. This means that $\mu_1(t) - r_1(t)\underline{1} \in \mathrm{Im}\ \sigma_1(t)$, and the thesis follows.

Theorem 2 If there are no arbitrage opportunities in the domestic country, $S_i(t)$, $r_i(t,x)$ and $X_{ij}(t)$ are solutions of equations (1), the process λ_1 is progressively measurable and the Novikov condition $\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \|\lambda_1(t)\|^2\, dt\right)\right] < +\infty$ holds, then there exists a probability $\mathbb{Q}_1$ equivalent to $\mathbb{P}$ under which the processes

$$\left(e^{-\int_t^T r_1(u)\ du} S_1(t)\right)_t \quad \textit{and} \quad \left(e^{-\int_t^T r_1(u)\ du} X_{1i}(t)S_i(t)\right)_t$$

are martingales. Moreover, under $\mathbb{P}$ we have:

$$\begin{aligned}
\mu_i(t) &= r_i(t,0) + \langle \sigma_i(t), \lambda_i(t)\rangle && (2)\\
\Gamma_i(t,T) &= -\int_0^{T-t} \tau_i(t,u)\ du && (3)\\
\alpha_i(t,\theta) &= \frac{\partial r_i(t,\theta)}{\partial \theta} - \tau_i(t,\theta)\Gamma_i(t,t+\theta) + \lambda_i(t) && (4)\\
m_{ij}(t) &= r_i(t) - r_j(t) + \sigma_{ij}^X(t)\lambda_i(t) && (5)\\
\sigma_{ij}^X(t) &= \lambda_i(t) - \lambda_j(t) && (6)
\end{aligned}$$

where $(\lambda_i(t))$ are the risk premium of the i-th economy and $(\Gamma_i(t,T))_t$ is the volatility of a zero coupon bond with maturity T in the i-th country. The probability $\mathbb{Q}_1$ is defined by the following Radon-Nikodym derivative with respect to $\mathbb{P}$:

$$\frac{d\mathbb{Q}_1}{d\mathbb{P}} = \exp\left(\int_0^T \langle \lambda_1(u), d\hat{W}(u)\rangle - \frac{1}{2}\int_0^T \|\lambda_1(u)\|^2\ du\right)$$

and the process

$$W_1(t) = \hat{W}(t) - \int_0^t \lambda_1(u)\ du$$

is a Brownian motion under $\mathbb{Q}_1$. Finally the equations (1) under $\mathbb{Q}_1$ can be rewritten as follows: $\forall i, j = 1, \ldots, n$

$$\begin{aligned}
dS_0(t) &= r_1(t)S_0(t)\ dt \\
dB_1(t,T) &= B_1(t,T)(r_1(t)\ dt + \Gamma_1(t,T)\ dW_1(t)) \\
dS_i(t) &= S_i(t)\left(\left(r_i(t) + \langle \sigma_i(t), \sigma_{i1}^X(t)\rangle\right)\ dt + \sigma_i(t)\ dW_1(t)\right) \\
dr_i(t,\theta) &= \left(\frac{\partial r_i}{\partial \theta}(t,\theta) - \tau_i(t,\theta)\Gamma_i(t,t+\theta) + \langle \tau_i(t,\theta), \sigma_{i1}^X(t)\rangle\right)\ dt + \tau_i(t,\theta)\ dW_1(t) \\
dX_{ij}(t) &= X_{ij}(t)\left(r_i(t) - r_j(t) + \langle \sigma_{ij}^X(t), \sigma_{i1}^X(t)\rangle\right)\ dt + \sigma_{ij}^X(t)\ dW_1(t)
\end{aligned}$$

Proof. The existence of $\mathbb{Q}_1$ follows from Novikov's condition and from Girsanov's theorem, as does the fact that $W_1(t)$ is a brownian motion under $\mathbb{Q}_1$. The existence of the risk premiums λ_i can be proved by standard absence of arbitrage arguments (the proof is similar to that of Lemma 1), so relation (2) is justified. Relation (3) can be derived by standard Itô calculus. Relation (4) can be derived by absence of arbitrage arguments (see the Musiela model in Musiela, 1993). Relation (5) and (6) are derived by arbitrage multicurrency arguments and are a straightforward generalization of the Garman-Kohlagen model in Garman and Kohlagen (1983).

We notice that if $\tau_i(\cdot)$ is a deterministic function, then $(r_i(t,\cdot))_t$ is a Markov process having values in the space $AC(\mathbb{R}^+, \mathbb{R})$, which can be identified with the Sobolev space $W^{1,1}_{\text{loc}}$; in the mathematical literature, to treat the problem more easily, it is preferred to suppose that $(r_i(t,\cdot))_t$ takes values in a separable Hilbert space H contained in $W^{1,1}_{\text{loc}}$ (for some possible examples, see Vargiolu, 1999). From this it follows that, if the σ_i, λ_i and τ_i are deterministic functions of the time for all $i = 1, \ldots, n$, then, for each choice of the maturity T, the process $(S_i(t), X_{ij}(t), r_i(t,\cdot), i, j = 1, \ldots, n)_t$ is a Markov process having values in $\mathbb{R}^n \times \mathbb{R}^{n\times n} \times H^n$.

3 The multiple option price

From now on, we'll make the assumption that the market is complete; this means that there exists only one risk neutral probability $\mathbb{Q}_1 \equiv \mathbb{P}$ such that the actualized prices of the assets in the domestic countries are (local) martingales, as seen in Theorem 2.

Now we want to evaluate a multiple European option having the final payoff

$$C_T = \sum_{j=1}^{N} (\langle \phi_j, \bar{S}(T)\rangle - K_j)\mathbf{1}_{\mathcal{E}_j}\ ,$$

where ϕ_j and K_j, $j = 1, \ldots, N$, are respectively n-dimensional real vectors and constants, and $\bar{S}_j(t) = X_{1j}(t)S_j(t)$ for $j = 2, \ldots, n$ and $t \in [0,T]$.

Theorem 3 If the Novikov conditions

$$\left.\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_1}\left[\exp\left(\frac{1}{2}\int_0^T \|\sigma_{1i}^X(t)\|^2\ dt\right)\right] < +\infty \\
&\mathbb{E}_{\mathbb{Q}_1}\left[\exp\left(\frac{1}{2}\int_0^T \|\sigma_i(t)\|^2\ dt\right)\right] < +\infty \\
&\mathbb{E}_{\mathbb{Q}_1}\left[\exp\left(\frac{1}{2}\int_0^T \|\Gamma_1(t,T)\|^2\ dt\right)\right] < +\infty
\end{aligned}\right\} \quad \forall i = 1, \ldots, n \tag{7}$$

are satisfied and the process $(S_i(t), X_{ij}(t), r_i(t,\cdot), i,j = 1,\dots,n)_t$ is Markov, then the price of the option having payoff C_T is given by

$$C(t) = \sum_{j=1}^{N}\left(\sum_{i=2}^{n}\phi_j^i X_{1i}(t)S_i(t)\mathbb{Q}_i^S(\mathcal{E}_j|\mathcal{F}_t) - B_1(t,T)K_j\mathbb{Q}_1^T(\mathcal{E}_j|\mathcal{F}_t)\right) \tag{8}$$

where $\mathbb{Q}_i^S$ is the risk neutral probability corresponding to the numeraire $\bar{S}_i$, and $\mathbb{Q}_1^T$ is the risk-neutral probability corresponding to the numeraire $B_1(t,T)$, better known as the *forward neutral probability* of the first country; $\mathbb{Q}_i^S$ and Q_1^T are defined by the following Radon-Nikodym derivatives:

$$\begin{aligned}
\frac{d\mathbb{Q}_i^S}{d\mathbb{Q}_1} &= \exp\left(\int_0^T (\langle\sigma_i(t) + \sigma_{1i}^X(t), dW_1(t)\rangle - \frac{1}{2}\int_0^T \|\sigma_i(t) + \sigma_{1i}^X(t)\|^2\, dt\right)\\
\frac{d\mathbb{Q}_1^T}{d\mathbb{Q}_1} &= \frac{e^{-\int_0^T r_1(u)\, du}}{B_1(0,T)} = \exp\left(\int_0^T \langle\Gamma_1(t,T), dW_1(t)\rangle - \frac{1}{2}\int_0^T \|\Gamma_1(t,T)\|^2\, dt\right)
\end{aligned}$$

Proof. Under the risk neutral probability $\mathbb{Q}_1$ the value in t of our multiple option is:

$$C(t) = \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\, du}\sum_{j=1}^{N}(\langle\phi_j, \bar{S}(T)\rangle - K_j)\mathbf{1}_{\mathcal{E}_j}\middle|\mathcal{F}_t\right]$$

We can linearize the payoff of the option, by using the exercise sets $\mathcal{E}_j$. Besides, we notice that under $\mathbb{Q}_1$ we have

$$\frac{d\bar{S}_j(t)}{\bar{S}_j(t)} = r_1(t)\, dt + \langle\sigma_j(t) + \sigma_{1j}^X(t), dW_1(t)\rangle \tag{9}$$

This gives us a formula for the price:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u)\, du}C_T\middle|\mathcal{F}_t\right] =\\
&= \sum_{j=1}^{N}\left(\mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u)\, du}\langle\phi_j, \bar{S}(T)\rangle\mathbf{1}_{\mathcal{E}_j}\middle|\mathcal{F}_t\right] - K_j\mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\, du}\mathbf{1}_{\mathcal{E}_j}\middle|\mathcal{F}_t\right]\right) =\\
&= \sum_{j=1}^{N}\left(\sum_{i=2}^{n}\mathbb{E}_{\mathbb{Q}_1}\left[\phi_j^i X_{1i}(t)S_i(t)\exp\left(\int_t^T \langle\sigma_i(u) + \sigma_{1i}^X(u), dW_1(u)\rangle +\right.\right.\right.\\
&- \left.\left.\left.\frac{1}{2}\int_t^T \|\sigma_i(u) + \sigma_{1i}^X(u)\|^2\, du\right)\mathbf{1}_{\mathcal{E}_j}\middle|\mathcal{F}_t\right] - K_j\mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\, du}\mathbf{1}_{\mathcal{E}_j}\middle|\mathcal{F}_t\right]\right) =\\
&= \sum_{j=1}^{N}\left(\sum_{i=2}^{n}\phi_j^i\bar{S}_i(t)\mathbb{E}_{\mathbb{Q}_i^S}[\mathbf{1}_{\mathcal{E}_j}|\mathcal{F}_t] - B_1(t,T)K_j\mathbb{E}_{\mathbb{Q}_1^T}[\mathbf{1}_{\mathcal{E}_j}|\mathcal{F}_t]\right) =\\
&= \sum_{j=1}^{N}\left(\sum_{i=1}^{n}\phi_j^i\bar{S}_i(t)\mathbb{Q}_i^S(\mathcal{E}_j|\mathcal{F}_t) - B_1(t,T)K_j\mathbb{Q}_1^T(\mathcal{E}_j|\mathcal{F}_t)\right)
\end{aligned}$$

The last member of the equality gives us the formula.

We have found a rather explicit formula for the price, which depends of the value of the traded assets $X_{1i}(t)S_i(t)$ at time t and of the exercise probabilities $\mathbb{Q}_i^S(\mathcal{E}_i|\mathcal{F}_t)$ and

$\mathbb{Q}_1^T(\mathcal{E}_1|\mathcal{F}_t))$ where we have written in different way the exercice set $\mathcal{E}_j$ because in the first case it is express in function of $\bar{S}_i(t)$ and in the second case in function of $S_1(t)$. Furthermore these probabilities depend on the particular dynamics of the forward rates in the n countries.

Now we show that, under the assumption that the forward rates processes are gaussian and the risk premiums λ_i of the n countries are deterministic, as the volatilities σ_i of S_i, we are able to derive a closed formula for the price of the option. To have gaussian forward rates processes is equivalent to have $\Gamma_i(t,T)$ deterministic. In this case it is rather easy to prove, under technical conditions, existence and uniqueness of the solution (in a weaker sense than the usual, namely in the mild sense, see Da Prato and Zabczyk, 1992) of the Musiela equation (as it is shown in Vargiolu, 1999).

Theorem 4 If $\Gamma_i(t,T)$, $\sigma_i(t)$ and $\sigma_{ij}^X(t)$ are deterministic functions belonging to $L^2([0,T])$ $\forall i,j=1,\ldots,n$, then the price of the currency multiple option is given by

$$\begin{aligned} C(t) &= \sum_{j=1}^N \left(\sum_{i=2}^n \phi_j^i X_{1i}(t) S_i(t) \mathbf{N}^{n-1}_{m_i(t),R_i(t)} \left\{ x \;\middle|\; \left(\frac{S_i(t)}{S_j(t)} e^{x_j} \right)_{j=1,\ldots,n,j\neq i} \in \mathcal{E}_i \right\} - \right. \\ &\left. - B_1(t,T) K_j \mathbf{N}^{n-1}_{m_1(t),R_1(t)} \left\{ x \;\middle|\; \left(\frac{S_1(t)}{S_j(t)} e^{x_j} \right)_{j=2,\ldots,n} \in \mathcal{E}_1 \right\} \right) \end{aligned} \tag{10}$$

where the vectors m_i are given by eq. (12), the matrices R_i are given by eq. (13), and $\mathbf{N}^{n-1}_{m,R}$ is the $(n-1)$-dimensional gaussian measure given by

$$\mathbf{N}^{n-1}_{m,R}(A) = \frac{1}{\sqrt{(2\pi)^{n-1}\det R}} \int_A e^{-\frac{1}{2}\langle R^{-1}(x-m),x-m\rangle}\, dx \quad \forall A \subseteq \mathbb{R}^{n-1} \text{ measurable} \tag{11}$$

where $d \in \mathbb{R}^{n-1}$ and R is a positive definite symmetric real $(n-1)\times(n-1)$ matrix.

Remark 5 From this theorem on a number of Gaussian probabilities of sets will appear. These sets will be often subsets of $\mathbb{R}^{\{1,\ldots,n\}\smallsetminus\{i\}}$, but for sake of simplicity we will always identify those sets as subsets of $\mathbb{R}^{n-1}$, though their elements would always be of the kind $x=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$.

Proof. If Γ_i, σ_i and σ_{ij}^X are $L^2([0,T])$ functions, then the Novikov conditions (7) are satisfied. Besides, since Γ_i, σ_i and σ_{ij}^X are deterministic, then the process $(S_i(t), X_{ij}(t), r_i(t,\cdot), i,j=1,\ldots,n)_t$ is Markov, so the formula (8) holds. In order to know the exercise probabilities, we have to write the dynamics of the assets $\bar{S}_j$ under the probabilities $\mathbb{Q}_i^S$ $\forall i,j=1,\ldots,n$, starting from (9).

Under $\mathbb{Q}_i^S$, the process $W_i^S(t) = W_1(t) - \int_0^t \left(\sigma_i(u) + \sigma_{1i}^X(u)\right)\, du$ is a brownian motion, and $\bar{S}_j$ has the dynamics

$$\begin{aligned} \frac{d\bar{S}_j(t)}{\bar{S}_j(t)} &= r_1(t,0)\, dt + \langle \sigma_j(t) + \sigma_{1j}^X(t), dW_i^S(t) + \left(\sigma_i(t) + \sigma_{1i}^X(t)\right)\, dt \rangle = \\ &= \left(r_1(t,0) + \langle \sigma_j(t) + \sigma_{1j}^X(t), \sigma_i(t) + \sigma_{1i}^X(t) \rangle \right) dt + \langle \sigma_j(t) + \sigma_{1j}^X(t), dW_i^S(t) \rangle \end{aligned}$$

Then we have:

$$\begin{aligned} \bar{S}_j(T) &= \bar{S}_j(t) \exp\left(\int_t^T \left(r_1(u,0) + \langle \sigma_{1j}^X(u) + \sigma_j(u), \sigma_{1i}^X(u) + \sigma_i(u) - \right.\right. \\ &\left.\left. - \frac{1}{2}\sigma_{1j}^X(u) - \frac{1}{2}\sigma_j(u) \rangle \right)\, du + \int_t^T \langle \sigma_j(u) + \sigma_{1j}^X(u), dW_i^S(u) \rangle \right) \end{aligned}$$

and

$$\begin{aligned}\frac{\bar{S}_i(T)}{\bar{S}_j(T)} &= \frac{\bar{S}_i(t)}{\bar{S}_j(t)} \exp\left(-\frac{1}{2}\int_t^T \left\|\sigma_i(u) - \sigma_j(u) + \sigma_{ji}^X(u)\right\|^2 \, du + \right.\\ &\left. + \int_t^T \langle \sigma_i(u) - \sigma_j(u) - \sigma_{ji}^X(u), dW_i^S(u)\rangle\right)\end{aligned}$$

Then we have found that the law of the vector $\left(\log\left(\frac{\bar{S}_i(T)/\bar{S}_i(t)}{\bar{S}_j(T)/\bar{S}_j(t)}\right)\right)_j$ under $\mathbb{Q}_i^S$ is:

$$\left(\log\left(\frac{\bar{S}_i(T)/\bar{S}_i(t)}{\bar{S}_j(T)/\bar{S}_j(t)}\right)\right)_{j=1,\dots,n} \sim N(m_i(t), R_i(t))$$

where $m_i(t)$ is a $(n-1)$-dimensional vector with the j-th component given by

$$m_i^j(t) = -\frac{1}{2}\int_t^T \left\|\sigma_i(u) - \sigma_j(u) - \sigma_{ji}^X(u)\right\|^2 \, du \tag{12}$$

and where the (j,k)-component of the matrix $R_i(t)$ is:

$$R_i^{jk}(t) = \int_t^T \left\langle \sigma_i(u) - \sigma_j(u) - \sigma_{ji}^X(u), \sigma_i(u) - \sigma_k(u) - \sigma_{ki}^X(u)\right\rangle \, du \tag{13}$$

So we can write the exercise risk neutral probability of i-th country:

$$\mathbb{Q}_i^S(\mathcal{E}_i) = \mathbf{N}^{n-1}_{m_i(t),R_i(t)}\left\{x \in \mathbb{R}^{n-1} \;\middle|\; \left(\frac{S_i(t)}{S_j(t)}e^{x_j}\right)_{j=1,\dots,n,j\neq i} \in \mathcal{E}_i\right\}$$

We have removed the conditioning because $(S_i(t), X_{ij}(t), r_i(t,\cdot), i,j = 1,\dots,n)_t$ is Markov.

In order to find the last exercise probability, we notice that under $\mathbb{Q}_1^T$, the process $W_1^T(t) = W_1(t) - \int_0^t \Gamma_1(u,T)\,du$ is a brownian motion, so the process $\bar{S}_j$ has the dynamics

$$\begin{aligned}\frac{d\bar{S}_j(t)}{\bar{S}_j(t)} &= r_1(t,0)\,dt + \left\langle \sigma_j(t) + \sigma_{1j}^X(t), dW_1^T(t) + \Gamma_1(t,T)\,dt\right\rangle = \\ &= \left(r_1(t,0) + \left\langle \sigma_{1j}^X(t) + \sigma_j(t), \Gamma_1(t,T)\right\rangle\right)\,dt + \left\langle \sigma_j(t) + \sigma_{1j}^X(t),\; dW_1^T(t)\right\rangle\end{aligned}$$

Then we have:

$$\begin{aligned}\bar{S}_j(T) &= \bar{S}_j(t)\exp\left(\int_t^T \left(r_1(u,0) + \langle \sigma_{1j}^X(u) + \sigma_j(u), \Gamma_1(u,T) - \frac{1}{2}\sigma_{1j}^X(u) - \right.\right.\\ &\left.\left. -\frac{1}{2}\sigma_j(u)\rangle\right)\,du + \int_t^T \langle \sigma_j(u) + \sigma_{1j}^X(u), dW_1^T(u)\rangle\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\bar{S}_1(T)}{\bar{S}_j(T)} &= \frac{\bar{S}_1(t)}{\bar{S}_j(t)}\exp\left(-\frac{1}{2}\int_t^T \|\sigma_j(u) - \Gamma_1(u,T) + \sigma_{1j}^X(u)\|^2\,du + \right.\\ &\left. + \int_t^T \langle \sigma_j(u) - \Gamma_1(u,T) + \sigma_{1j}^X(u), dW_i^T(u)\rangle\right)\end{aligned}$$

so we can find the last exercise probability:

$$\mathbb{Q}_1^T(\mathcal{E}_1) = \mathbf{N}^{n-1}_{m_1(t),R_1(t)} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; \left(\frac{S_1(t)}{S_j(t)} e^{x_j}\right)_{j=2,\dots,n} \in \mathcal{E}_1 \right\}$$

where

$$m_1^j(t) = \frac{1}{2} \int_t^T \|\sigma_j(u) - \Gamma_1(u,T) + \sigma_{1j}^X(u)\|^2 \; du \tag{14}$$

4 The hedging strategy

We have found the price of the currency multiple option of the form:

$$\begin{aligned} C(t) &= \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\;du} \left(\sum_{j=1}^N (\langle \phi_j, \bar{S}(T)\rangle - K_j)\mathbf{1}_{\mathcal{E}_j} \right) \middle| \mathcal{F}_t \right] = \\ &= \sum_{j=1}^N \left(\sum_{i=2}^n \mathbb{E}_{\mathbb{Q}_i^S}\left[\phi_j^i X_{1i}(t) S_i(t) \mathbf{1}_{\mathcal{E}_i} | \mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}_1^T}\left[K_j B_1(t,T)\mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] \right) \end{aligned}$$

Now we want to find a hedging portfolio in terms of the assets $\bar{S}_i(t) = X_{1i}(t)S_i(t)$, $i = 1,\dots,n$, that is we want to build a self-financing portfolio

$$V(t) = \sum_{i=1}^n H_i(t) X_{1i}(t) S_i(t)$$

such that $V(t) = C(t)\ \forall t \leq T$ $\mathbb{Q}$-a.s.

Theorem 6 Under the same assumptions of theorem 3, the hedging portfolio is given by:

$$\begin{aligned} H_1(t) &= -\sum_{j=1}^N K_j \mathbb{Q}_1^T(\mathcal{E}_1|\mathcal{F}_t) \\ H_k(t) &= \mathbb{Q}_k^S(\mathcal{E}_k|\mathcal{F}_t) \cdot \sum_{j=1}^N \phi_j^k \quad \forall k = 2,\dots,n \end{aligned}$$

Proof. We may write the price of the option as a deterministic function of $\bar{s}_1 = \bar{S}_1(t) = S_1(t)$ and $\bar{s}_i = \bar{S}_i(t) = X_{1i}(t)S_i(t)\ \forall i = 2,\dots,n$ as follows:

$$F(\bar{s}_1,\dots,\bar{s}_n) = \sum_{j=1}^N \left(\sum_{i=2}^n \mathbb{E}_{\mathbb{Q}_i^S}\left[\phi_j^i \bar{s}_i \mathbf{1}_{\mathcal{E}_i}|\mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}_1^T}\left[K_j \bar{s}_1 \mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] \right)$$

The proportion of the i-th asset $\bar{S}_i$ in the replicating self financing portfolio is, as it is well known (see for example Lamberton and Lapeyre, 1991), the derivative of the function $F(\bar{s}_1,\dots,\bar{s}_n)$ with respect to $\bar{s}_i$, so we have:

$$\frac{\partial F}{\partial \bar{s}_k} = \sum_{j=1}^N \left(\sum_{i=2}^n \mathbb{E}_{\mathbb{Q}_i^S}\left[\frac{\partial}{\partial \bar{s}_k} \phi_j^i \bar{s}_i \mathbf{1}_{\mathcal{E}_i} \middle| \mathcal{F}_t \right] - \mathbb{E}_{\mathbb{Q}_1^T}\left[\frac{\partial}{\partial \bar{s}_k} K_j \bar{s}_1 \mathbf{1}_{\mathcal{E}_j} \middle| \mathcal{F}_t \right] \right)$$

so:

$$\begin{aligned}
\frac{\partial F}{\partial \bar{s}_1} &= -\sum_{j=1}^{N} K_j \mathbb{E}_{\mathbb{Q}_1^T}\left[\mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] = -\sum_{j=1}^{N} K_j \mathbb{Q}_1^T(\mathcal{E}_1|\mathcal{F}_t) \\
\frac{\partial F}{\partial \bar{s}_k} &= \sum_{j=1}^{N} \phi_j^k \mathbb{Q}_k^S(\mathcal{E}_k|\mathcal{F}_t) = \mathbb{Q}_k^S(\mathcal{E}_k|\mathcal{F}_t) \sum_{j=1}^{N} \phi_j^k \qquad \forall k = 2, \ldots, n
\end{aligned}$$

where we have taken the derivative under the expectation sign and where the terms with $i \neq k$ in the sum and the last one are null because their integrands are 0 almost everywhere (with respect to the different $\mathbb{Q}_i^S$ and $\mathbb{Q}_1^T$).

5 Johnson's option on the maximum of several assets

Now we present a particular case, namely Johnson's option on the maximum of several assets. This is an option on several assets in the same market, and it is a famous case of option of several assets; here we see that Johnson's formula can be derived as a straightforward application of ours.

In this case our assets are all traded in a single market so all the countries $1, \ldots, n$ coincide, and we have $X_{1i} \equiv 1$ $\forall i = 1, \ldots, n$. Besides the interest rate is supposed to be deterministic and having a flat structure: $r_1(t,x) \equiv r$, so $\sigma_{1i}^X \equiv 0$, $\Gamma_i \equiv 0$ and $\mathbb{Q}_1^T = \mathbb{Q}_1$ $\forall T > 0$. For this reason there is no need to consider the dynamics of the zero coupon bond with maturity T (because it coincides with the money market account), so we change the notation and suppose that S_1 is the money market account and $S_2, \ldots, S_n$ are risky assets without qualitative differences between them. Finally, the assets evolve according to equations having deterministic time-independent coefficients: $\sigma_i(t) \equiv \sigma_i$. We notice that

$$\frac{d\mathbb{Q}_i^S}{d\mathbb{Q}_1} = \exp\left(\int_0^T \langle\sigma_i, dW_1(t)\rangle\rangle - \frac{1}{2}\int_0^T \|\sigma_i\|^2 \, dt\right) = \exp\left(\langle\sigma_i, W_1(T)\rangle\rangle - \frac{1}{2}\|\sigma_i\|^2 T\right)$$

The cash flow at maturity T of the option on the maximum of the assets $S_1, \ldots, S_n$ with maturity T and strike price K is:

$$C_T = (S_{\max}(T) - K)^+$$

where

$$S_{\max}(t) = \max_{j=2,\ldots,n} S_j(t)$$

We can linearize the payoff of the option, by introducing the following exercise sets, in a slightly different way than the ones of the general option;

$$\begin{aligned}
\mathcal{E}_i &= \{S_i(T) \geq S_j(T) \ \forall j \neq i\} \quad \forall i = 2, \ldots, n \\
\mathcal{E}_1 &= \left\{\max_{2 \leq j \leq n} S_j(T) \geq K\right\}
\end{aligned}$$

These sets have an appealing intuitive meaning: in fact, the set $\mathcal{E}_0$ represents the possibility to exercise the option, and the sets $\mathcal{E}_i$, $i = 2, \ldots, n$, represent the choice of the i-th asset for the payment. We notice that the sets $\mathcal{E}_i$, $i = 2, \dots, n$ are mutually disjoint, so we can write the option on the maximum in this way:

$$\begin{aligned} C_T &= (S_{\max}(T) - K)^+ = \left(\max_i S_i(T) - K\right) \mathbf{1}_{\mathcal{E}_1} = \\ &= \left(\sum_{i=2}^n S_i(T)\mathbf{1}_{\mathcal{E}_i} - K\right) \mathbf{1}_{\mathcal{E}_1} = \sum_{i=2}^n S_i(T)\mathbf{1}_{\mathcal{E}_i \cap \mathcal{E}_1} - K\mathbf{1}_{\mathcal{E}_1} \end{aligned}$$

Proposition 7 (Johnson's formula) Under the assumptions above, the price of the option on the maximum of several assets is given by

$$C(t) = \sum_{i=2}^n S_i(t) N_{n-1}(d_i, \bar{R}_i) - Ke^{-r(T-t)}(1 - N_{n-1}(d_0, \bar{R}_0))$$

where the vectors d_i are given by eq. (17), the matrixes $\bar{R}_i$ are given by eq. (16), and N_{n-1} is the cumulative distribution function of a $(n-1)$-dimensional gaussian law:

$$N_{n-1}(d, \bar{R}) = \frac{1}{\sqrt{(2\pi)^{n-1} \det R}} \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} \mathbf{1}_{\{x_i \le d_i\}} e^{-\frac{1}{2}\langle \bar{R}^{-1}x, x\rangle} \, dx \tag{15}$$

The hedging portfolio is given by

$$\begin{cases} H_0(t) = 1 - N_{n-1}(d_1, \bar{R}_1) \\ H_k(t) = N_{n-1}(d_i, \bar{R}_i) \quad \forall k = 1, \dots, n \end{cases}$$

Proof. The law of the vector $\left(\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right)\right)_{j \ne i}$ under $\mathbb{Q}_i^S$ is:

$$\left(\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right)\right)_{j \ne i} \sim N(m_i(t), R_i(t))$$

where $m_i(t)$ is a $(n-1)$-dimensional vector with the following j-th component:

$$m_i^j(t) = -\frac{1}{2} \int_t^T \|\sigma_i - \sigma_j\|^2 \, du = -\frac{1}{2}\|\sigma_i - \sigma_j\|^2 (T-t)$$

and where the (j,k)-component of the matrix $R_i(t)$ is:

$$R_i^{jk}(t) = \int_t^T \langle \sigma_i - \sigma_j, \sigma_i - \sigma_k \rangle \, du = \langle \sigma_i - \sigma_j, \sigma_i - \sigma_k \rangle (T-t)$$

So we can write the exercise probability of the i-th asset:

$$\begin{aligned} Q_i^S(\mathcal{E}_i \cap \mathcal{E}_1) = \mathbb{Q}_i^S \Bigg\{ &\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right) \ge \log\left(\frac{S_j(t)}{S_i(t)}\right) \ \forall j \ne i, \\ &\log\left(\frac{S_i(T)\,/S_i(t)}{e^{-r(T-t)}}\right) \ge \log\left(\frac{Ke^{-r(T-t)}}{S_i(t)}\right) \Bigg\} = \\ = \mathbb{Q}_i^S &\left\{ \frac{\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right) - m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \ge \frac{\log\left(\frac{S_j(t)}{S_i(t)}\right) - m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \ \forall j \ne i, \right. \end{aligned}$$

$$\frac{\log\left(\frac{S_i(T)/S_i(t)}{e^{-r(T-t)}}\right) - m_i^1(t)}{\sqrt{R_i^{11}(t)}} \geq \frac{\log\left(\frac{Ke^{-r(T-t)}}{S_i(t)}\right) - m_i^1(t)}{\sqrt{R_i^{11}(t)}} \Bigg\} =$$

$$= \mathbf{N}_{0,\bar{R}_i}^{n-1} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; x_j \leq \frac{\log\left(\frac{S_i(t)}{S_j(t)}\right) + m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \;\forall j \neq i, x_1 \leq \frac{\log\left(\frac{S_i(t)}{Ke^{-r(T-t)}}\right) + m_i^1(t)}{\sqrt{R_i^{11}(t)}} \right\}$$

where

$$\bar{R}_i^{jk}(t) = \frac{R_i^{jk}(t)}{\sqrt{R_i^{jj}(t) R_i^{kk}(t)}} \tag{16}$$

This leads to the following:

$$\mathbb{Q}_i^S(\mathcal{E}_i \cap \mathcal{E}_0) = N_{n-1}(d_i, \bar{R}_i)$$

where d_i is a $(n-1)$-dimensional vector with components

$$d_i^j = \frac{\log\left(\frac{S_i(t)}{S_j(t)}\right) + m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \quad \forall j \neq 1, i, \qquad d_i^1 = \frac{\log\left(\frac{S_i(t)}{Ke^{-r(T-t)}}\right) + m_i^1(t)}{\sqrt{R_i^{11}(t)}} \tag{17}$$

We can find the last exercise probability:

$$\begin{aligned}
\mathbb{Q}_1^T(\mathcal{E}_1) &= 1 - \mathbb{Q}_1^T\{S_j(T) \leq K \quad \forall j \neq 1\} = \\
&= 1 - \mathbb{Q}_1^T \left\{ \frac{S_j(T)/S_j(t)}{e^{-r(T-t)}} \leq \frac{Ke^{-r(T-t)}}{S_j(t)} \;\forall j \neq 1 \right\} = \\
&= 1 - \mathbb{Q}_1^T \left\{ \frac{\log\left(\frac{e^{-r(T-t)}}{S_j(T)/S_j(t)}\right) - m_1^j(t)}{\sqrt{R_1^{jj}(t)}} \geq \frac{\log\left(\frac{S_j(t)}{Ke^{-r(T-t)}}\right) - m_1^j(t)}{\sqrt{R_1^{jj}(t)}} \;\forall j \neq 1 \right\} = \\
&= 1 - \mathbf{N}_{0,\bar{R}_1}^{n-1} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; x_i \leq \frac{\log\left(\frac{Ke^{-r(T-t)}}{S_j(t)}\right) + m_1^j(t)}{\sqrt{R_1^{jj}(t)}} \;\forall j \neq 1 \right\} = \\
&= 1 - \mathbf{N}_{n-1}(d_1, \bar{R}_1)
\end{aligned}$$

where $\bar{R}_1$ is given by equation (16), and d_1 is a $(n-1)$-dimensional vector with components

$$d_1^j = \frac{\log\left(\frac{Ke^{-r(T-t)}}{S_j(t)}\right) + m_1^j(t)}{\sqrt{R_1^{jj}(t)}}, \quad j \neq 1$$

The composition of the hedging strategy follows immediately from theorem 6.

We notice that we have obtained Johnson's formula for the evaluation of the option; this formula is already present in Johnson (1987), and so the hedging strategy is implicit in his work, though it is presented without a rigorous proof.

6 The MAP strategy

Here we present another example of application of our derivative asset, namely Fong-Vasicek's MAP (Multiple Asset Performance) strategy Fong and Vasicek (1989). This strategy allows us to obtain the best performance of stock indexes S_i of several countries, provided a certain price (that is the price of the option on the maximum) is paid. Hence this strategy has the final payoff:

$$\mathrm{MAP}(T) = \cdot \max_{i=1,\ldots,n} \left(\frac{X_{1i}(T)S_i(T)}{X_{1i}(0)S_i(0)} \right) = \max_{i=1,\ldots,n} (K_i X_{1i}(T) S_i(T)) \tag{18}$$

where $X_{11} \equiv 1$ and the K_i are fixed at the beginning of the contract in this way:

$$K_i = \frac{1}{X_{1i}(0)S_i(0)}$$

The choice of the K_i is motivated by the fact that the percentage performance of the i-th asset over the period $[0,T]$ is $\frac{X_{1i}(T)S_i(T)}{X_{1i}(0)S_i(0)}$.

In this section, we do not suppose that $S_1(\cdot) = B_1(\cdot, T)$; this is because the final payoff is a function only of the values of n assets compared between them, and does not depend on a fixed quota (like for example in the option on the maximum), so as we will see, $B(\cdot, T)$ will not appear neither in the pricing formula nor in the hedging strategy.

The sets $\mathcal{E}_i$ in this case are:

$$\mathcal{E}_i = \{K_i X_{1i}(T)S_i(T) \geq K_j X_{1j}(T) S_j(T) \ \forall j \neq i\} \quad \forall i = 1, \ldots, n$$

These sets have an appealing intuitive meaning: in fact, the set $\mathcal{E}_i$, $i = 1, \ldots, n$, represent the choice of the i-th asset. We notice that the sets $\mathcal{E}_i$, $i = 1, \ldots, n$ are mutually disjoint, so we can write the MAP as

$$C_T = S_{\max}(T) = \sum_{i=1}^{n} K_i X_{1i}(T) S_i(T) \mathbf{1}_{\mathcal{E}_i}$$

So we get the result:

Proposition 8 Under the same assumptions of theorem 4, the price of the MAP strategy is given by

$$C(t) = C \sum_{i=1}^{n} \frac{X_{1i}(t)S_i(t)}{X_{1i}(0)S_i(0)} N_{n-1}(d_i, \bar{R}_i) \tag{19}$$

where d_i are given by eq. (20, $\bar{R}_i$ are given by eq. (16), and N_{n-1} is given by (15); it can be hedged by a portfolio $(H_1(t), \ldots, H_n(t))_t$ in the assets $\bar{S}_1, \ldots, \bar{S}_n$, given by:

$$H_i(t) = K_i N_{n-1}(d_i, \bar{R}_i)$$

Proof. We can write the exercise risk neutral probability of i-th country:

$$\begin{aligned}
\mathbb{Q}_i^S(\mathcal{E}_i) &= \mathbb{Q}_i^S \left\{ \frac{K_i \bar{S}_i(T)}{K_j \bar{S}_j(T)} \geq 1, \ j \neq i \right\} = \\
&= \mathbb{Q}_i^S \left\{ \log \left(\frac{\bar{S}_i(T)/\bar{S}_i(t)}{\bar{S}_j(T)/\bar{S}_j(t)} \right) \geq \log \left(\frac{K_j \bar{S}_j(t)}{K_i \bar{S}_i(t)} \right) \ \forall j \neq i \right\} =
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{Q}_i^S \left\{ \frac{\log\left(\frac{\bar{S}_i(T)/\bar{S}_i(t)}{\bar{S}_j(T)/\bar{S}_j(t)}\right) - m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \geq \frac{\log\left(\frac{K_j\bar{S}_j(t)}{K_i\bar{S}_i(t)}\right) - m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \ \forall j \neq i \right\} = \\
&= \mathbf{N}_{0,\bar{R}_i}^{n-1} \left\{ x \in \mathbb{R}^{n-1} \ \middle| \ x_j \leq \frac{\log\left(\frac{K_i\bar{S}_i(t)}{K_j\bar{S}_j(t)}\right) + m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \ \forall j \neq i \right\}
\end{aligned}
$$

So if we define the $(n-1)$-dimensional vector d_i as

$$
d_i^j = \frac{\log\left(\frac{K_i\bar{S}_i(t)}{K_j\bar{S}_j(t)}\right) + m_i^j(t)}{\sqrt{R_i^{jj}(t)}} \quad \forall j \neq i \tag{20}
$$

then we have the first part of the proposition. The composition of the hedging strategy follows immediately from theorem 6.

7 Options on the arithmetic mean

Here we present a third example of application, that is a call option on the arithmetic mean of $n-1$ assets. This example appears when we have options on a stock index that is an arithmetic mean of the most significative assets in the market (for example, MIB30 in Italy, CAC40 in France, S&P500 in the U.S.A.), even if, in the case n is too big, calculation difficulties can arise.

The final payoff is:

$$
C_T = \left(\sum_{j=2}^n a_j S_j(T) - K \right)^+ = \left(\sum_{j=2}^n a_j S_j(T) - K \right) \mathbf{1}_{\mathcal{E}_1}
$$

where

$$
\mathcal{E}_1 = \left\{ \sum_{j=2}^n a_j S_j(T) - K \geq 0 \right\}
$$

The main difficulty in this case is that in general a sum of lognormal random variables is not lognormal, so it is not possible to express the price of the option using simple functions (for example, via a one-dimensional gaussian distribution); anyway we can express it via a cumulative $(n-1)$-dimensional gaussian distribution.

Proposition 9 Under the same assumptions of theorem 3, the price of the call option on the arithmetic mean is

$$
\begin{aligned}
C(t) &= \sum_{j=2}^n a_j S_j(t) \mathbf{N}_{m_i(t),R_i(t)}^{n-1} \left\{ x \in \mathbb{R}^{n-1} \ \middle| \ \sum_{j=1, j\neq i}^n a_j S_j(t) e^{x_j} \geq K e^{-x_i} \right\} + \\
&\quad -KB_1(t,T) \mathbf{N}_{m_1(t),R_1(t)}^{n-1} \left\{ x \in \mathbb{R}^{n-1} \ \middle| \ \sum_{j=2}^n a_j S_j(t) e^{x_j} \geq KB_1(t,T) \right\}
\end{aligned}
$$

and the hedging strategy is given by a portfolio $(H_1(t),\ldots,H_n(t))_t$ in the assets $S_1,\ldots,S_n$ so that

$$\begin{aligned}
H_j(t) &= a_j \mathbf{N}^{n-1}_{m_i(t),R_i(t)} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; \sum_{j=1,j\neq i}^{n} a_j S_j(t) e^{x_j} \geq K e^{-x_i} \right\} \\
H_0(t) &= -K \mathbf{N}^{n-1}_{m_1(t),R_1(t)} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; \sum_{j=2}^{n} a_j S_j(t) e^{x_j} \geq K B_1(t,T) \right\}
\end{aligned}$$

Proof. We can write the exercise probability under the measure $\mathbb{Q}_i^S$:

$$\begin{aligned}
\mathbb{Q}_i^S(\mathcal{E}_i) &= \mathbb{Q}_i^S \left\{ \sum_{j=2}^{n} a_j S_j(T) \geq K \right\} = \mathbb{Q}_i^S \left\{ \sum_{j=2}^{n} a_j S_j(t) \frac{S_j(T)}{S_j(t)} \geq K \right\} = \\
&= \mathbf{N}^{n-1}_{m_i(t),R_i(t)} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; \sum_{j=1,j\neq i}^{n} a_j S_j(t) e^{x_j} \geq K e^{-x_i} \right\}
\end{aligned}$$

where $m_i(t)$ and $R_i(t)$ are given by (12) and (13); besides the last exercise probability is

$$\begin{aligned}
\mathbb{Q}_1^T(\mathcal{E}_1) &= \mathbb{Q}_1^T \left\{ \sum_{j=2}^{n} a_j S_j(T) \geq K \right\} = \mathbb{Q}_1^T \left\{ \sum_{j=2}^{n} a_j S_j(t) \frac{S_j(T)}{S_j(t)} \geq K B_1(t,T) \right\} = \\
&= \mathbf{N}^{n-1}_{m_1(t),R_1(t)} \left\{ x \in \mathbb{R}^{n-1} \;\middle|\; \sum_{j=2}^{n} a_j S_j(t) e^{x_j} \geq K B_1(t,T) \right\}
\end{aligned}$$

where $m_1(t)$ is given by (14). The composition of the hedging portfolio follows from theorem 5.

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RIASSUNTO IN ITALIANO

Il nostro scopo é proporre una strategia di valutazione e di replicazione per un tipo generale di opzione su piú titoli in un mondo internazionale multidivisa senza arbitraggi con tassi di interesse gaussiani. Come caso particolare viene derivata la formula di Johnson per l'opzione sul massimo di piú titoli, e vengono presentati due esempi di applicazione, in particolare la strategia MAP e l'opzione sulla media aritmetica di diversi titoli.

RIASSUNTO IN INGLESE

Our aim is to propose an evaluation and a replicating strategy for a general kind of multiasset option in an international multicurrencies no-arbitrage world with Gaussian interest rates. Johnson's formula for the option on the maximum of several assets is derived as a particular case of ours, and two examples of application, namely the MAP strategy and the option on the arithmetic mean of several assets, are presented.

Prezzaggio ed "hedging" di un tipo generale di opzione su piú titoli