Pricing and hedging of the currency multiple option on the maximum of several bonds

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Abstract

Our aim is to propose an evaluation and a replicating strategy for a currency multiple option in an international multicurrencies no-arbitrage world with Gaussian interest rates. An example of application, namely the MAP strategy, is presented.

1 Introduction

In this paper we present a closed formula for the price and for the replicating strategy of the currency multiple option having the payoff

$$C_T = \left(\max_{j=2,\ldots,n} K_j S_j(T) - 1 \right)^+ ,$$

where the S_j are the values $B_j(t,T)$ of zero-coupon bonds of the foreign country j at time t, each one having maturity T, converted in domestic currency via the exchange rate $X_{1j}(t)$ at time t, that is

$$S_j(t) = X_{1j}(t) B_j(t,T) \text{ for } j = 2,3,\ldots,n \text{ and } t \in [0,T]$$

This multiple currency option, which is called option on the maximum of several bonds, is an option on the maximum of n currencies, the domestic one and $n-1$ foreign ones. In our model, besides the assumption of absence of arbitrage opportunities, we make the further hypotheses that the quantities significant to our analysis (that is the prices $B_j(t,T)$ in t of the zero coupon bonds of the j-th country maturing in T, the exchange rates $X_{ij}(t)$ between the i-th country and the j-th one, and the spot forward rate $r_j(t,x)$ in time t with maturity $t+x$ of the j-th country) satisfy stochastic differential equations driven by a k-dimensional Wiener process. This leads us to a specific structure for the stochastic differential equations satisfied by these quantities. If we make the further natural assumption that the quantities cited above considered as a whole process is Markov, then we are able to linearize the price of the multiple option and to get a formula in terms of a linear combination of the assets S_j, weighted by the probabilities of suitable exercise sets. In order to arrive to such a formula, we change the numeraire in each of the elements of the linear combination using in each country j first the corresponding risk-neutral probability $\mathbb{Q}_j$, and then the forward-neutral probability $\mathbb{Q}_j^T$, introduced in [7] and [16]. With the additional assumptions that the risk premium and the diffusion term of the forward rates are deterministic in all the countries, we are able to derive explicit formulae both for the price and for the hedging portfolio of the multiple option. Finally we present an application of our option, the MAP strategy (Multiple Asset Performance), presented in [8].

This work uses three different topics in finance: international finance, term structure of interest rates and options on several assets, focusing much on the third topic. The first relevant work in international finance is the one of Garman-Kohlagen [9], which is the seminal paper in the subject in the same sense as the paper of Black-Scholes [3] is in pricing and hedging of European options. The main idea presented in the Garman-Kohlagen model is the assumption of absence of arbitrage opportunities between the countries, formalized by the fact that the foreign prices expressed in terms of the domestic currency by the exchange rate behaves as a domestic price. Through this approach we may see the exchange rate as a domestic asset that pays a continous dividend which corresponds to the foreign interest rate. Term structure of interest rates is a wide topic in finance, but here we don't concentrate much on it. We only notice that, since we are dealing with bonds and exchange rates, we have to suppose that the interest rates in our n countries are stochastic. In order to derive explicit formulae, though, we suppose that the volatility of the bonds and the risk premiums in all the n countries are deterministic. A deterministic volatility of the bonds is equivalent to the hypothesis that the forward rates are Gaussian processes (see for example [1], [15] and [20]). In particular, we use the Musiela model (see [4], [20] or [22]) for the interest rates rather than the Heath-Jarrow-Morton (HJM) model (see [15]), because the first easily allows one to see that the instantaneous forward rates $(r_i(t, \cdot))_t$ of the different countries $i = 1, \dots, n$ are Markov processes. Coming to the third topic involved in this work, the first work about options on several assets is Margrabe's paper [19], that analyses the simplest case of an option to exchange one asset for another. Later, Stulz [21] arrived to price an option on the maximum of two assets. His work was generalized by Johnson [17], who solved the same problem for a general number of assets. The first and the last work follow more or less the same idea, that is to linearize the payoff function and to find closed formulas in terms of a ponderate sum of different probabilities calculated in different exercise sets in terms of the multivariate Gaussian distribution function. In particular we follow Johnson's technique and find an expression similar to Johnson's formula. This also allows us to find a replicating strategy in the n assets: this derives intuitively from Johnson's formula, even if Johnson himself doesn't derive the replication strategy from it. Moreover in this work there is the further complication of stochastic interest rates. Because of the Gaussian model we have chosen, though, this complication affects the general structure of the price and replication formulae only in terms of more complicated coefficients than Johnson's ones.

The paper is organized as follows: in section 2 we present the model we used; in section 3 we derive the pricing formula for the multiple exchange option; in section 4 we derive the hedging strategy from the pricing formula we have found, and in section 5 we present an example of application of our option, namely Fong-Vasicek's MAP strategy [8].

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2 The model

We consider n countries, each one having a different currency. We assume that country 1 is our domestic country and that the countries indicated with the numbers $2, 3, \ldots, n$ are foreign ones. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we represent the information at time $t \in [0, T]$ with a filtration $(\mathcal{F}_t)_t$, which for technical purposes we assume to be complete and right continuous. We also suppose that all the processes are adapted to the filtration $(\mathcal{F}_t)_t$. We name:

$r_i(t,\theta)$ the instantaneous forward rate prevailing at time t for the maturity $t+\theta = T$ in the i-th country,

$B_i(t,T)$ the price at time t of a zero coupon bond with maturity T in the i-th country,

$X_{ij}(t)$ the exchange rate from the i-th country to the j-th country, that is the price of 1 unity of the j-th currency expressed in i-th currency.

We suppose that the dynamic of the processes under the historic probability $\mathbb{P}$ are:

$$\begin{array}{rcl} dB_i(t,T) & = & B_i(t,T)(\mu_i(t,T)\ dt + \Gamma_i(t,T)\ d\hat{W}(t)) \\ dr_i(t,\theta) & = & \alpha_i(t,\theta)\ dt + \tau_i(t,\theta)\ d\hat{W}(t) \\ dX_{ij}(t) & = & X_{ij}(t)(m(t)\ dt + \sigma^X_{ij}(t)\ d\hat{W}(t)) \end{array} \tag{1}$$

where $(\hat{W}(t))_t$ is a k-dimensional Brownian motion which represent the sources of risk which affect the different economies, and where we suppose that all the quantities we introduced satisfy the technical regularity conditions for the integrals to be defined.

We suppose there are no arbitrage opportunities in the domestic market; we shall see in the next theorem that this implies the existence of a probability measure $\mathbb{Q}_1$ equivalent to $\mathbb{P}$, under which the actualized prices of all the domestic assets are martingales, and that there are several contraints on the quantities introduced before.

Theorem 1 *If there are no arbitrage opportunities in the domestic country, and $B_i(t,T)$, $r_i(t,x)$ and $X_{ij}(t)$ are solutions of Equations (1), then there exists a probability $\mathbb{Q}_1$ equivalent to $\mathbb{P}$ under which the processes*

$$\left(e^{-\int_t^T r_1(u,0)\ du} B_1(t,T)\right)_t \quad \text{and} \quad \left(e^{-\int_t^T r_1(u,0)\ du} X_{1i}(t) B_i(t,T)\right)_t$$

are martingales. Moreover, under $\mathbb{P}$ we have:

$$\mu_i(t,T) = r_i(t,0) + \langle \Gamma_i(t,T), \lambda_i(t) \rangle \tag{2}$$

$$\Gamma_i(t,T) = -\int_0^{T-t} \tau_i(t,u)\ du \tag{3}$$

$$\alpha_i(t,\theta) = \frac{\partial r_i(t,\theta)}{\partial \theta} - \tau_i(t,\theta)\Gamma_i(t,t+x) + \lambda_i(t) \tag{4}$$

$$m_{ij}(t) = r_i(t,0) - r_j(t,0) + \sigma^X_{ij}(t)\lambda_i(t) \tag{5}$$

$$\sigma^X_{ij}(t) = \lambda_i(t) - \lambda_j(t) \tag{6}$$

where $(\lambda_i(t))$ are the risk premium of the i-th economy. The probability $\mathbb{Q}_1$ is defined by the following Radon-Nikodym derivative with respect to $\mathbb{P}$:

$$\frac{d\mathbb{Q}_1}{d\mathbb{P}} = \exp\left(\int_0^T \lambda_1(u)\ d\hat{W}(u) - \frac{1}{2}\int_0^T \|\lambda_1(u)\|^2\ du\right)$$

and the process

$$W_1(t) = \hat{W}(t) - \int_0^t \lambda_1(u)\ du$$

is a Brownian motion under $\mathbb{Q}_1$. Finally the dynamics of the previous processes under $\mathbb{Q}_1$ can be rewritten as follows: $\forall i, j = 1, \ldots, n$

$$\begin{aligned}
dB_i(t,T) &= B_i(t,T)\left(\left(r_i(t,0) + \langle \Gamma_i(t,T), \sigma_{i1}^X(t)\rangle\right)\ dt + \Gamma_i(t,T)\ dW_1(t)\right) \\
dr_i(t,\theta) &= \left(\frac{\partial r_i}{\partial \theta}(t,\theta) - \tau_i(t,\theta)\Gamma_i(t,t+\theta) + \langle \tau_i(t,\theta), \sigma_{i1}^X(t)\rangle\right)\ dt + \tau_i(t,\theta)\ dW_1(t) \\
dX_{ij}(t) &= X_{ij}(t)\left(r_i(t,0) - r_j(t,0) + \langle \sigma_{ij}^X(t), \sigma_{i1}^X(t)\rangle\right) + \sigma_{ij}^X(t)\ dW_1(t)
\end{aligned}$$

Proof. The existence of $\mathbb{Q}_1$ follows from Novikov's condition and from Girsanov's theorem, as does the fact that $W(t)$ is a Brownian motion under $\mathbb{Q}_1$. The existence of the risk premiums λ_i can be proved by standard absence of arbitrage arguments (see [6], [7]), so relation (2) is justified. Relation (3) can be derived by standard Itô calculus. Relation (4) can be derived by absence of arbitrage arguments (see the Musiela model in [20]). Relation (5) and (6) are derived by arbitrage multicurrency arguments and are a straightforward generalization of the Garman-Kohlagen model in [9]. $\diamond$

We notice that if $\tau_i(\cdot)$ is a deterministic function, then $(r_i(t,\cdot))_t$ is a Markov process having values in the space $AC(\mathbb{R}^+, \mathbb{R})$, which can be identified with the Sobolev space $W^{1,1}_{\mathrm{loc}}$; in the mathematical literature, to treat the problem more easily, it is preferred to suppose that $(r_i(t,\cdot))_t$ takes values in a separable Hilbert space H contained in $W^{1,1}_{\mathrm{loc}}$ (for some possible examples, see [10] and [22]). From this it follows that, if the Γ_i, λ_i and τ_i are deterministic functions of the time for all $i = 1, \ldots, n$, then, for each choice of the maturity T, the process $(B_i(t,T), X_{ij}(t), r_i(t,\cdot), i, j = 1, \ldots, n)_t$ is a Markov process having values in $\mathbb{R}^n \times \mathbb{R}^{n\times n} \times H^n$.

3 The currency multiple option price

Now we want to evaluate a currency multiple option on the maximum of several bonds, all having the same maturity T. This asset gives the right to exchange at the maturity T of all the bonds the (unitary) payoff of a domestic bond with the payoff of K_i shares of a foreign (unitary) bond of the country i converted in the domestic currency, for $i = 2, \ldots, n$; here the number K_i is fixed at the beginning of the contract, and represents the ratio between the desired final payoff from the i-th country and the exchange rate in T expected in 0. These K_i are fixed in the contract at time 0. For a possible specific choice, corresponding to the MAP strategy, we send the reader to section 5.

The cash flow at maturity T of a currency multiple option with maturity T and strike price K on the maximum of n currencies is:

$$C_T = (S_{\max}(T) - 1)^+$$

where

$$\begin{aligned}
S_j(t) &= X_{1j}(t)B_j(t,T) \text{ for } j = 2, 3, \ldots, n \text{ and } t \in [0,T] \\
S_{\max}(t) &= \max(K_i S_j(t), j = 2, \ldots, n)
\end{aligned}$$

Theorem 2 . *If the Novikov conditions*

$$\begin{cases} \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\sigma_{1i}^X(t)\|^2\ dt\right)\right] < +\infty \\ \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\Gamma_i(t,T)\|^2\ dt\right)\right] < +\infty \end{cases} \quad \forall i = 1,\dots,n \tag{7}$$

are satisfied and the process $(B_i(t,T), X_{ij}(t), r_i(t,\cdot), i,j = 1,\dots,n)_t$ is Markov, then the price of the option having payoff C_T is given by

$$\begin{aligned} C(t) &= \sum_{i=2}^{n} K_i X_{1i}(t) B_i(t,T) \mathbb{Q}_i^T \left\{ \frac{X_{1i}(T)}{X_{1j}(T)} \geq 1,\ j \neq 1,i,\ K_i X_{1i}(T) \geq 1 \right\} - \\ &\quad - B_1(t,T) \mathbb{Q}_1^T \left\{ \max_j K_j X_{1j}(T) B_j(T,T) \geq 1 \right\} \end{aligned} \tag{8}$$

where $\mathbb{Q}_i$ and $\mathbb{Q}_i^T$ are respectively the risk neutral probabilities and the forward neutral probabilities of the i-th country, defined by the following Radon-Nikodym derivatives:

$$\frac{d\mathbb{Q}_i}{d\mathbb{Q}_1} = \exp\left(\int_0^T \sigma_{1i}^X(t)\ dW_1(t) - \frac{1}{2}\int_0^T \|\sigma_{1i}^X(t)\|^2\ dt\right)$$

$$\frac{d\mathbb{Q}_i^T}{d\mathbb{Q}_i} = \frac{e^{-\int_0^T r_i(u,0)du}}{B_i(0,T)} = \exp\left(\int_0^T \Gamma_i(t,T)\ dW_i(t) - \frac{1}{2}\int_0^T \|\Gamma_i(t,T)\|^2\ dt\right)$$

Proof. Under the risk neutral probability $\mathbb{Q}_1$ the value in t of our multiple option is:

$$C(t) = \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du}(S_{\max}(T) - K)^+ | \mathcal{F}_t\right]$$

We can linearize the payoff of the option, by introducing the following exercise sets:

$$\begin{aligned} \mathcal{E}_i &= \{K_i X_{1i}(T) B_i(T,T) \geq K_j X_{1j}(T) B_j(T,T)\ \forall j \neq 1, i\} \quad \forall i = 2,3,...,n \\ \mathcal{E}_1 &= \left\{ \max_{1 \leq j \leq n} K_j X_{1j}(T) B_j(T,T) \geq B_1(T,T) \right\} \end{aligned}$$

These sets have an appealing intuitive meaning: in fact, the set $\mathcal{E}_1$ represents the possibility to exercise the option, and the sets $\mathcal{E}_i$, $i = 2,3,\dots,n$, represent the choice of the i-th currency for the payment. We notice that the sets $\mathcal{E}_i$, $i = 2,3,\dots,n$ are mutually disjoint, so we can write the exchange option as

$$\begin{aligned} C_T &= (S_{\max}(T) - K)^+ = \left(\max_i (K_i X_{1i}(T) B_i(T,T)) - 1\right) \mathbf{1}_{\mathcal{E}_1} = \\ &= \left(\sum_{i=2}^n K_i X_{1i}(T) B_i(T,T) \mathbf{1}_{\mathcal{E}_i} - 1\right) \mathbf{1}_{\ \mathcal{E}_1} = \sum_{i=2}^n K_i X_{1i}(T) B_i(T,T) \mathbf{1}_{\mathcal{E}_i \cap \mathcal{E}_1} - \mathbf{1}_{\mathcal{E}_1} \end{aligned}$$

This gives us a formula for the price:

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du} C_T | \mathcal{F}_t\right] = \\ &\quad = \mathbb{E}_{\mathbb{Q}_1}\left[\sum_{i=2}^n e^{-\int_t^T r_1(u,0)\ du} K_i X_{1i}(T) B_i(T,T)\ \mathbf{1}_{\mathcal{E}_i \cap \mathcal{E}_1} | \mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du} \mathbf{1}_{\mathcal{E}_1} | \mathcal{F}_t\right] = \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}_1}\left[\sum_{i=2}^{n} K_i X_{1i}(t)\exp\left(-\int_t^T r_i(u,0)\ du - \int_t^T \sigma_{1i}^X\ dW_1(u) + \right.\right.\\
&\left.\left. + \frac{1}{2}\int_t^T |\sigma_{1i}^X(u)|\ du\right)\mathbf{1}_{\mathcal{E}_i\cap\mathcal{E}_1}|\mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du}\mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] = \\
&= \sum_{i=2}^{n} K_i X_{1i}(t)\mathbb{E}_{\mathbb{Q}_i}\left[e^{-\int_t^T r_i(u,0)\ du}\mathbf{1}_{\mathcal{E}_i\cap\mathcal{E}_1}|\mathcal{F}_t\right] - \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du}\ \mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] = \\
&= \sum_{i=2}^{n} K_i X_{1i}(t)B_i(t,T)\mathbb{E}_{\mathbb{Q}_i^T}\left[\mathbf{1}_{\mathcal{E}_i\cap\mathcal{E}_1}|\mathcal{F}_t\right] - B_1(t,T)\mathbb{E}_{\mathbb{Q}_1^T}\left[\mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] = \\
&= \sum_{i=2}^{n} K_i X_{1i}(t)B_i(t,T)\mathbb{Q}_i^T\left\{\mathcal{E}_i\cap\mathcal{E}_1\right\} - B_1(t,T)\mathbb{Q}_1^T\left\{\mathcal{E}_1\right\}
\end{aligned}
$$

where we have removed the conditioning because $(B_i(t,T), X_{ij}(t), r_i(t,\cdot), i,j = 1,\ldots,n)_t$ is Markov. The last member of the equality gives us the thesis. $\diamond$

We have found a rather explicit formula for the price, which depends of the value of the traded assets $X_{1i}(t)B_i(t,T)$ at time T and of the exercise probabilities $\mathbb{Q}_i^T\{\mathcal{E}_i\cap\mathcal{E}_1\}$ and $\mathbb{Q}_1^T\{\mathcal{E}_1\}$. Furthermore these probabilities depend on the particular dynamics of the forward rates in the n countries.

Now we show that, under the assumption that the forward rates processes are Gaussian and the risk premiums λ_i of the n countries are deterministic, we are able to derive a closed formula for the price of the option. To have Gaussian forward rates processes is equivalent to have deterministic bond volatilities $\Gamma_i(t,T)$. In this case it is rather easy to prove, under technical conditions, existence and uniqueness of the solution (in a weaker sense than the usual, namely in the mild sense, see [5]) of the Musiela equation (as it is shown in [22]).

Theorem 3 . *If $\Gamma_i(t,T)$ and $\sigma_{ij}^X(t)$ are deterministic functions belonging to $L^2([0,T])$ $\forall i,j = 1,\ldots,n$, then the price of the currency multiple option is given by*

$$
C(T) = \sum_{i=1}^{n} K_i X_{1i}(t)B_i(t,T)N_{n-1}(d_i,\bar{R}_i) - B_1(t,T)(1 - N_{n-1}(d_1,\bar{R}_1)) \tag{9}
$$

where the vectors d_i and the matrixes $\bar{R}_i$ will be given later, and N_{n-1} is the cumulative distribution function of a $(n-1)$-dimensional Gaussian law:

$$
N_{n-1}(d,\bar{R}) = \frac{1}{(\sqrt{2\pi})^{n-1}}\int_{\mathbb{R}^{n-1}}\prod_{i=1}^{n-1}\mathbf{1}_{\{x_i\le d_i\}}e^{-\frac{1}{2}<\bar{R}^{-1}x,x>}\ dx
$$

Proof. If Γ_i and σ_{ij}^X are $L^2([0,T])$ functions, then the Novikov conditions (7) are satisfied. Besides, since Γ_i and σ_{ij}^X are deterministic, the the process $(B_i(t,T), X_{ij}(t), r_i(t,\cdot), i,j = 1,\ldots,n)_t$ is Markov, so the formula (8) holds. In order to know the exercise probabilities, we write the dynamics of the assets s_j under the probabilities $\mathbb{Q}_i^T$ $\forall i,j = 1,\ldots,n$.

Under the domestic risk neutral probability the process S_j has the dynamics

$$
dS_j(t) = X_{1j}(t)\ dB_j(t,T) + B_j(t,T)\ dX_{1j}(t) + d\langle B_j(\cdot,T), X_{1j}(\cdot)\rangle_t
$$

then we have:

$$
\frac{dS_j(t)}{S_j(t)} = r_1(t,0)\ dt + \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)\ dW_1(t)
$$

Under the risk neutral probability of the i-th country $\mathbb{Q}_i$, the process $W_i(t) = W_1(t) - \int_0^t \sigma_{1i}^X(u)\, du$ is a Brownian motion, and the process S_j has the dynamics

$$\begin{aligned}\frac{dS_j(t)}{S_j(t)} &= r_1(t,0)\; dt + \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)(dW_i(t) + \sigma_{1i}(t)\; dt) = \\ &= \left(r_1(t,0) + \sigma_{1j}^X(t)\sigma_{1i}^X(t) + \Gamma_j(t,T)\sigma_{1i}(t)\right)\; dt + \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)\; dW_i(t)\end{aligned}$$

Under the forward neutral probability of i-th country $\mathbb{Q}_i^T$, the process $W_i^T(t) = W_i(t) - \int_0^t \Gamma_i(u,T)\, du$ is a Brownian motion, and the process S_j has the dynamics

$$\begin{aligned}\frac{dS_j(t)}{S_j(t)} &= \left(r_1(t,0) + \sigma_{1j}^X(t)\sigma_{1i}^X(t) + \Gamma_j(t,T)\sigma_{1i}(t)\right)\; dt + \\ &\quad + \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)\left(dW_i^T(t) + \Gamma_i(t,T)\; dt\right) = \\ &= \left(r_1(t,0) + (\sigma_{1j}^X(t) + \Gamma_j(t,T))(\sigma_{1i}^X(t) + \Gamma_i(t,T))\right)\; dt + \\ &\quad + \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)\; dW_i^T(t)\end{aligned}$$

We can write explicitely $S_j(T)$ under the probability $\mathbb{Q}_i^T$:

$$\begin{aligned}S_j(T) &= S_j(t)\exp\left(\int_t^T \left(r_1(t,0) + (\sigma_{1j}^X(t) + \Gamma_j(t,T))(\sigma_{1i}^X(t) + \Gamma_i(t,T))\right)\; dt + \right.\\ &\quad \left. + \int_t^T \left(\Gamma_j(t,T) + \sigma_{1j}^X(t)\right)\; dW_i^T(t) - \frac{1}{2}\int_t^T \left\|\Gamma_j(t,T) + \sigma_{1j}^X(t)\right\|^2\; dt\right)\end{aligned}$$

Then we have:

$$\begin{aligned}\frac{S_i(T)}{S_j(T)} &= \frac{S_i(t)}{S_j(t)}\exp\left(\int_t^T \frac{1}{2}\left(\left\|\Gamma_j(t,T) + \sigma_{1j}^X(t)\right\|^2 + \left\|\Gamma_j(t,T) + \sigma_{1j}^X(t)\right\|^2\right) - \right.\\ &\quad -(\sigma_{1j}^X(t) + \Gamma_j(t,T))(\sigma_{1i}^X(t) + \Gamma_i(t,T))\; du + \\ &\quad \left. + \int_t^T (\Gamma_i(u,T) - \Gamma_j(u,T) - \sigma_{ji}^X(u))\; dW_i^T(u)\right)\end{aligned}$$

Then we have found that the law of the vector $\left(\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right)\right)_j$ under $\mathbb{Q}_i^T$ is:

$$\left(\log\left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)}\right)\right)_{j=1,\dots,n} \sim N(m_i(t), R_i(t))$$

where $m_i(t)$ is a $(n-1)$-dimensional vector with the j-th component given by

$$\begin{aligned}m_i^j(t) &= \int_t^T \left(\frac{1}{2}\left(\left\|\Gamma_j(t,T) + \sigma_{1j}^X(t)\right\|^2 + \left\|\Gamma_j(t,T) + \sigma_{1j}^X(t)\right\|^2\right) - \right.\\ &\quad \left. -(\sigma_{1j}^X(t) + \Gamma_j(t,T))(\sigma_{1i}^X(t) + \Gamma_i(t,T))\right)\; du\end{aligned}$$

and where the (j,k)-component of the matrix $R_i(t)$ is:

$$R_i^{jk}(t) = \int_t^T (\Gamma_i(u,T) - \Gamma_j(u,T) - \sigma_{ji}^X(u))(\Gamma_i(u,T) - \Gamma_k(u,T) - \sigma_{ki}^X(u))\; du$$

So we can write the exercise forward neutral probability of i-th country:

$$\begin{aligned}
\mathbb{Q}_i^T(\mathcal{E}_i \cap \mathcal{E}_1) &= \mathbb{Q}_i^T \left\{ \frac{K_i S_i(T)}{K_j S_j(T)} \geq 1, \; j \neq 1, i, \; K_i S_i(T) \geq S_1(T) \right\} = \\
&= \mathbb{Q}_i^T \left\{ \log \left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)} \right) \geq \log \left(\frac{K_j S_j(t)}{K_i S_i(t)} \right) \; \forall j \neq 1, i, \right. \\
&\quad \left. \log \left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)} \right) \geq \log \left(\frac{S_1(t)}{K_i S_i(t)} \right) \right\} = \\
&= \mathbb{Q}_i^T \left\{ \frac{\log \left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)} \right) - m_{ij}(t)}{\sqrt{R_i^{jj}(t)}} \geq \frac{\log \left(\frac{K_j S_j(t)}{K_i S_i(t)} \right) - m_{ij}(t)}{\sqrt{R_i^{jj}(t)}} \; \forall j \neq 1, i, \right. \\
&\quad \left. \frac{\log \left(\frac{S_i(T)/S_i(t)}{S_j(T)/S_j(t)} \right) - m_{i1}(t)}{\sqrt{R_i^{11}(t)}} \geq \frac{\log \left(\frac{S_1(t)}{K_i S_i(t)} \right) - m_{i1}(t)}{\sqrt{R_i^{11}(t)}} \right\} = \\
&= N_{n-1}(0, \bar{R}_i) \left\{ x_j \leq \frac{\log \left(\frac{K_i S_i(t)}{K_j S_j(t)} \right) + m_{ij}(t)}{\sqrt{R_i^{jj}(t)}} \; \forall j \neq 1, i, \right. \\
&\quad \left. x_1 \leq \frac{\log \left(\frac{K_i S_i(t)}{K S_1(t)} \right) + m_{i1}(t)}{\sqrt{R_i^{11}(t)}} \right\}
\end{aligned}$$

where

$$\bar{R}_i^{jk}(t) = \frac{R_i^{jk}(t)}{\sqrt{R_i^{jj}(t) R_i^{kk}(t)}} \tag{10}$$

and $N_{n-1}(0, \bar{R}_i)$ is an $(n-1)$-dimensional Gaussian law with mean 0 and covariance matrix $\bar{R}_i$. This leads to

$$\mathbb{Q}_i^T(\mathcal{E}_i \cap \mathcal{E}_1) = N_{n-1}(d_i, \bar{R}_i) \; ,$$

where N_{n-1} is the cumulative distribution function of a $(n-1)$-dimensional Gaussian law:

$$N_{n-1}(d, \bar{R}) = \frac{1}{(\sqrt{2\pi})^{n-1}} \int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} \mathbf{1}_{\{x_i \leq d_i\}} e^{-\frac{1}{2} < \bar{R}^{-1} x, x >} \; dx$$

and d_i is a $(n-1)$-dimensional vector with components

$$d_{ij} = \frac{\log \left(\frac{K_i X_{1i}(t) B_i(t,T)}{K_j X_{1j}(t) B_j(t,T)} \right) + m_{ij}(t)}{\sqrt{R_i^{jj}(t)}} \quad \forall j \neq 1, i$$

$$d_{iK} = \frac{\log \left(\frac{K_i X_{1i}(t) B_i(t,T)}{B_1(t,T)} \right) + m_{i1}(t)}{\sqrt{R_i^{11}(t)}}$$

We can find the last exercise probability:

$$\mathbb{Q}_1^T(\mathcal{E}_1) = 1 - \mathbb{Q}_1^T \{ K_j S_j(T) \leq K S_1(T) \quad \forall j \neq 1 \} =$$

$$
\begin{aligned}
&= 1-\mathbb{Q}_1^T\left\{\frac{S_j(T)/S_j(t)}{S_1(T)/S_1(t)} \le \frac{S_1(t)}{K_jS_j(t)}\ \forall j\neq 1\right\} = \\
&= 1-\mathbb{Q}_1^T\left\{\frac{\log\left(\frac{S_1(T)/S_1(t)}{S_j(T)/S_j(t)}\right)-m_{1j}(t)}{\sqrt{R_1^{jj}}} \ge \frac{\log\left(\frac{K_jS_j(t)}{S_1(t)}\right)-m_{1j}(t)}{\sqrt{R_1^{jj}}}\ \forall j\neq 1\right\} = \\
&= 1-N_{n-1}(0,\bar{R}_1)\left\{x_i \le \frac{\log\left(\frac{S_1(t)}{K_jS_j(t)}\right)+m_{1j}(t)}{\sqrt{R_1^{jj}}}\ \forall j\neq 1\right\} = \\
&= 1-N_{n-1}(d_1,\bar{R}_1)
\end{aligned}
$$

where $\bar{R}_1$ is given by equation (10), and d_1 is a $(n-1)$-dimensional vector with components

$$
d_{1j} = \frac{\log\left(\frac{B_1(t,T)}{K_jX_{1j}(t)B_j(t,T)}\right)+m_{1j}(t)}{\sqrt{R_1^{jj}(t)}}
$$

$\diamond$

4 The hedging strategy

We have found the price of the currency multiple option of the form:

$$
\begin{aligned}
C(t) &= \mathbb{E}_{\mathbb{Q}_1}\left[e^{-\int_t^T r_1(u,0)\ du}\left(\sum_{i=2}^n K_iX_{1i}(T)B_i(T,T)\mathbf{1}_{\mathcal{E}_i}-1\right)\mathbf{1}_{\mathcal{E}_1}|\mathcal{F}_t\right] = \\
&= \sum_{i=2}^n \mathbb{E}_{\mathbb{Q}_i^T}\left[K_iX_{1i}(t)B_i(t,T)\mathbf{1}_{\mathcal{E}_i\cap\mathcal{E}_1}\right]-\mathbb{E}_{\mathbb{Q}_1^T}\left[B_1(t,T)\mathbf{1}_{\mathcal{E}_1}\right]
\end{aligned}
$$

Now we want to find a hedging portfolio in terms of the assets $S_i(t) = X_{1i}(t)B_i(t,T)$, $i = 1,\dots,n$, that is we want to build a self-financing portfolio

$$
V(t) = \sum_{i=1}^n H_i(t)X_{1i}(t)B_i(t,T)
$$

such that $V(t) = C(t)\ \forall t \le T$ $\mathbb{Q}$-a.s.

Theorem 4 . *The hedging portfolio is given by:*

$$
\begin{cases}
H_1(t) = -\mathbb{Q}_1^T(\mathcal{E}_1) = 1-N_{n-1}(d_1,\bar{R}_1) \\
H_i(t) = K_i\mathbb{Q}_i^T(\mathcal{E}_i\cap\mathcal{E}_1) = K_iN_{n-1}(d_i,\bar{R}_i) \quad \forall i = 2,\dots,n
\end{cases}
$$

Proof. We may write the price of the option as a deterministic function of $s_1 = S_1(t,T) = B_1(t,T)$ and $s_i = S_i(t,T) = X_{1i}(t,T)B_i(t,T)\ \forall i = 2,\dots,n$ as follows:

$$
F(s_1,\dots,s_n) = \sum_{i=2}^n \mathbb{E}_{\mathbb{Q}_i^T}\left[K_is_i\mathbf{1}_{\mathcal{E}_i(s)\cap\mathcal{E}_1(s)}\right]-\mathbb{E}_{\mathbb{Q}_i^T}\left[s_1\mathbf{1}_{\mathcal{E}_1(s)}\right]
$$

The proportion of the i-th asset S_i in the replicating self financing portfolio is, as it is well known (see for example [18]), the derivative of the function $F(s_1, \ldots, s_n)$ with respect to s_i, so we have:

$$\frac{\partial F}{\partial s_i} = \sum_{i=2}^{n} \mathbb{E}_{\mathbb{Q}_j^T} \left[\frac{\partial}{\partial s_i} K_j s_j \mathbf{1}_{\mathcal{E}_i(s) \cap \mathcal{E}_1(s)} \right] - \mathbb{E}_{\mathbb{Q}_1^T} \left[\frac{\partial}{\partial s_j} s_1 \mathbf{1}_{\mathcal{E}_i(s)} \right]$$

so:

$$\frac{\partial F}{\partial s_1} = -\mathbb{E}_{\mathbb{Q}_1^T}[\mathbf{1}_{\mathcal{E}_1}] = -\mathbb{Q}_1^T(\mathcal{E}_1)$$

$$\frac{\partial F}{\partial s_i} = K_i \mathbb{E}_{\mathbb{Q}_i^T} \left[\mathbf{1}_{\mathcal{E}_i(s) \cap \mathcal{E}_1(s)} \right] = K_i \mathbb{Q}_i^T(\mathcal{E}_i(s) \cap \mathcal{E}_1(s)) \quad \forall i = 2, \ldots, n$$

where we have taken the derivative under the expectation sign and where the terms with $i \neq j$ in the sum and the last one are null because their integrands are 0 almost everywhere (with respect to the different $\mathbb{Q}_i^T$). $\diamond$

5 An example of application: the MAP strategy

Here we present an example of application of the option on the maximum, namely Fong-Vasicek's [8] MAP (Multiple Asset Performance) strategy. This strategy allows us to obtain the best performance of several assets, which in our case are bonds of different countries, provided a certain price (that is the price of the option on the maximum) is paid. Hence this strategy has the final payoff:

$$MAP(T) = \max_{i=1,\ldots,n} (X_{1i}(T) B_i(T,T) K_i) \tag{11}$$

where $X_{11} \equiv 1$, $K_1 = 1$ and K_i are fixed at the beginning of the contract in this way:

$$K_i = \frac{B_1(0,T)}{X_{1i}(0) B_i(0,T)}$$

To see that this is the right choice of the K_i, we present the following argument: we can place an amount of money $B_1(0,T)$ in the domestic economy by buying a bond with maturity T, obtaining the payoff 1 at the maturity; else we can invest the same amount of money, which is equal to $B_1(0,T)X_{i1}(0)$ shares of the i-th currency, in the i-th country by buying a bond with maturity T, obtaining the payoff $B_1(0,T)X_{i1}(0)/B_i(0,T)$ shares of the i-th currency at the maturity, that is $X_{1i}(T)B_1(0,T)/(X_{1i}(0)B_i(0,T))$; so we can see that the MAP strategy has the final payoff given by Eq. (11). We notice that we may decompose this payoff in this way:

$$MAP(T) = B_1(T,T) + \left(\max_{i=1,2,\ldots,n} K_i(X_{1i}(T) B_i(T,T) - B_1(T,T) \right)^+ \tag{12}$$

so we can write $MAP(T) = B_1(T,T) + C(T)$, where $C(T)$ is our option on the maximum. So we get the result:

Proposition 5 . *The price in t of the MAP strategy is given by:*

$$MAP(t) = B_1(t,T) + C(t)$$

portfolio $(H_1(t), \ldots, H_n(t))_t$ *in the assets* $S_1, \ldots, S_n$*, given by:*

$$\begin{aligned} H_1(t) &= 1 + 1 - N_{n-1}(d_1, \bar{R}_1) = 2 - N_{n-1}(d_i, \bar{R}_i) \\ H_i(t) &= K_i N_{n-1}(d_i, \bar{R}_i) \quad \forall i = 2, \ldots, n \end{aligned}$$

Proof. It follows immediately from the decomposition (12). ◇

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