Optimal portfolio in a regime-switching model

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Abstract

In this paper we derive the solution of the classical Merton problem, i.e. maximizing the utility of the terminal wealth, in the case when the risky assets follow a diffusion model with switching coefficients. We find out that the optimal portfolio is a generalisation of the corresponding one in the classical Merton case, with portfolio proportions which depend on the market regime. We perform our analysis via the classical approach with the Hamilton-Jacobi-Bellman equation. First we extend the mutual fund theorem as present in [5] to our framework. Then we show explicit solutions for the optimal strategies in the particular cases of exponential, logarithm and power utility functions.

1 Introduction

The standard model for a financial market is the following. Let the bond price B_t satisfy

$$dB_t = rB_t dt,$$

with $r > 0$ deterministic, and the stock prices $S_t = (S_t^1, \dots, S_t^d)$ satisfy

$$dS_t = \text{diag }(S_t)(\mu(t, S_t)\ dt + \Sigma(t, S_t)\ dW_t), \tag{1}$$

where $\mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are suitable functions and W is a d-dimensional Brownian motion defined on a suitable probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Assume

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also that Σ has full rank. In this situation the classical problems (utility maximization, pricing and hedging of derivatives) are solved.

However, these kinds of model fail in incorporating sudden changes on the dynamics of the assets, occurring for example during a financial crisis. Among the various models which are more suitable for this, we will analyse the so-called regime-switching models, introduced for the first time in [12]: while these models are now broadly used in various fields of financial mathematics (see e.g. [1, 2, 3, 4, 6, 7, 10, 14, 17, 22] and references therein), it seems that the classical problem of utility maximisation of terminal wealth has not been yet addressed in this framework.

To fix the ideas, let us consider the classical Merton problem: assume that $\mu_t \equiv \mu$ and $\Sigma_t \equiv \Sigma$ are deterministic and known, and an agent wants to maximise his(her) expected utility from terminal wealth

$$\mathbb{E}[U(X_T)]$$

where the self-financing portfolio X has dynamics

$$dX_t = X_t\left[(h_t \cdot \mu + (1 - h_t \cdot \mathbf{1})r)\,dt \;+\; h_t \Sigma dW_t\right]$$

with h_t^i the proportion of wealth invested in the i-th risky asset, $i = 1, \ldots, d$, and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$. It is well known (see for example [5]) that, if for example $U(x) = x^\gamma/\gamma$, then the optimal portfolio allocation is given by the constant proportions

$$\hat{h}_t \equiv h := \frac{1}{1-\gamma}(\Sigma\Sigma^T)^{-1}(\mu - r\mathbf{1})$$

Thus, no matter what happens in the world, an investor would always try to keep these portfolio proportions. A natural question arises: is this world too simple?

The idea of regime-switching models is that the "economy" can assume $m \geq 2$ different states, and when there is a change of state, prices change dynamics. In this simple example, coefficients μ and Σ depend on this state. As reported in [15], the typical intuitive situation for this is the following: the economy switches between $m = 2$ states, i.e. "business as usual" (BAU, state 1) and "crisis" (state 2), with the typical following stylized facts:

- (much) higher variances in Σ_2 than in Σ_1;
- significantly larger correlations from Σ_2 than from Σ_1, reflecting contagion effects;
- predominantly negative μ_2 $(-r\mathbf{1})$, reflecting down market effects.

The way how this could impact the classical Merton problem is seen in the following numerical example.

Example 1.1. *Assume $d = 2$ and that in the BAU state both assets have yield equal 0.01 $+r$ and volatility 0.2, with correlation 0.1; this can be represented as $\mu_1 - r\mathbf{1} = (0.01, 0.01)$ and*

$$\Sigma_1\Sigma_1^T = \begin{pmatrix} 0.04 & 0.004 \\ 0.004 & 0.04 \end{pmatrix}$$

For an agent with risk-aversion coefficient $\gamma = 0.1$, this gives the optimal portfolio

$$h = \frac{1}{1-0.1}(\Sigma_1\Sigma_1^T)^{-1}(\mu_1 - r\mathbf{1}) \simeq (0.2525, 0.2525)$$

so this agent will invest the 25.25% of its wealth in each of the two risky assets.

Assume now that in the "crisis" state assets do not change their yield, but the two volatilities increase to 0.3 and the correlation to 0.6. Thus, now we still have $\mu_2 - r\mathbf{1} = (0.01, 0.01)$ *and*

$$\Sigma_2\Sigma_2^T = \begin{pmatrix} 0.09 & 0.054 \\ 0.054 & 0.09 \end{pmatrix}$$

For the same agent than before, this gives the optimal portfolio

$$h = \frac{1}{1-0.1}(\Sigma_2\Sigma_2^T)^{-1}(\mu_2 - r\mathbf{1}) \simeq (0.0771, 0.0771)$$

so, under the "crisis" state, the agent will diminish his(her) investment down to 7.71% in each of the two risky assets.

The aim of this paper is to present a way to make the above argument (and optimal portfolios) rigorous. In particular, in Section 2 we will define the regime-switching model, frame the utility maximization problem in this context and present a classical way to solve it, namely the dynamic programming approach with Hamilton-Jacobi-Bellman (HJB) equation, which in this framework becomes a system of differential equations. In Section 3 we will derive a mutual fund theorem, which generalizes the classical one holding in diffusion models (see e.g. [5]). In Sections 4, 5 and 6, respectively, we analyze more in detail the classical cases of exponential, logarithm and power utility functions, generalizing classical results.

2 The model

As in [4], we begin by assuming that the bond price B_t satisfies

$$dB_t = rB_t dt,$$

for some $r > 0$, and that there are $m \in \mathbb{N}$ states of the world and d (non-defaultable) risky assets, with values in $D := (0, \infty)^d$. Let the stock prices $S_t = (S_t^1, \ldots, S_t^d)$ satisfy

$$\begin{cases} dS_t &= \text{diag }(S_t)(\mu_{\eta_{t-}}(t, S_t)\ dt + \Sigma_{\eta_{t-}}(t, S_t)\ dW_t), \\ d\eta_t &= \sum_{k,j=1}^m (j-k)\mathbb{I}_{\{k\}}(\eta_{t-})dN_t^{kj}, \end{cases} \tag{2}$$

where, for each $i = 1, \ldots, m$, $\mu_i : [0,T] \times D \to \mathbb{R}^d$, $\Sigma_i : [0,T] \times D \to \mathbb{R}^{d\times d}$ are functions such that $(t, s) \to \text{diag }(s)\mu_i(t, s)$ and $(t, s) \to \text{diag }(s)\Sigma_i(t, s)$ are C^1 on $[0,T] \times D$, W is a Brownian motion and $N = (N^{kj})_{1\le k,j\le m}$ is a multivariate $\mathbb{F}$-adapted point process such that

$$(N_t^{kj})_t \text{ has } (P, \mathbb{F})\text{-intensity } \lambda^{kj}(t, S_t) \tag{3}$$

with bounded C^1 functions $\lambda^{kj} : [0,T] \times D \longrightarrow [0, \infty)$; W and N are independent and are both defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. This implies (see [4]) that for all $(t, s, k) \in [0,T] \times D \times \{1, \ldots, m\}$ there is a unique strong solution (S, η) to Equation (2) starting from $(S_t, \eta_t) = (s, k)$, up to a possibly finite random explosion time. Thus, we also assume that

$$\mathbb{P}\{S_u \in D \text{ for all } u \in [t,T]\} = 1 \qquad \text{for all } (t, s, k) \in [0,T] \times D \times \{1, \ldots, m\} \tag{4}$$

Remark 2.1. *As in [14] (and implicitly in [2, 3, 4, 7, 8]), we assume that the process η is observable: in fact, if we assume that the Σs are distinct, then the local quadratic variation-covariation of the risky assets S in any small interval to the left of t will yield $\Sigma_{\eta_{t-}}$ exactly. Hence, even if S is not Markovian, (S, η) is jointly so.*

We now build a self-financing portfolio with initial capital $X_0 > 0$, with the following dynamics:

$$dX_t = \left(\pi_t \cdot \mu_{\eta_{t-}}(t, S_t) + (X_t - \pi_t \cdot \mathbf{1})r\right)\ dt + \pi_t \Sigma_{\eta_{t-}}(t, S_t) dW_t \tag{5}$$

where

$$\pi_t^i = S_t^i \theta_t^i \tag{6}$$

is the wealth invested in the i-th risky asset, with θ_t^i being the number of i-th stocks in hand at time t. An alternative definition of X, more usual when we have the additional constraint $X_t > 0$ (for example when dealing with logarithmic or power utility function) is the following:

$$dX_t = X_t \left[\left(h_t \cdot \mu_{\eta_{t-}}(t, S_t) + (1 - h_t \cdot \mathbf{1})r\right)\ dt + h_t \Sigma_{\eta_{t-}}(t, S_t) dW_t\right] \tag{7}$$

where

$$h_t^i = \frac{S_t^i \theta_t^i}{X_t} = \frac{\pi_t^i}{X_t} \tag{8}$$

is, as in the Introduction, the proportion of wealth invested in the i-th risky asset, with θ_t^i being the number of i-th stocks in hand at time t.

We define

$$J(t, x, s, \eta; \pi) := \mathbb{E}[U(X_T^{t,x,s,\eta;\pi})]$$

and the *value function*

$$V(t, x, s, \eta) = \sup_{\pi \in \Theta} J(t, x, s, \eta; \pi). \tag{9}$$

where $(X^{t,x,s,\eta;h}, S^{t,x,s,\eta;h}, \eta^{t,x,s,\eta;h})$ is the 3-dimensional controlled Markov process starting from (x, s, η) at time t with the dynamics defined by Equations (2) and (5) with the control $\pi \in \Theta[t, T]$, where $\Theta[t, T]$ is the set of *admissible* controls, i.e. predictable processes on $[t, T]$ such that Equation (5) has a unique strong solution $X^{t,x,s,\eta;\pi}$ for each initial condition (x, s, η) at time t such that $\mathbb{E}[U(X_T^{t,x,s,\eta;\pi})] \in \mathbb{R}$, $(e^{-ru} X_u^\pi)_u \in M^2([t, T])$ and $(\pi_u U(X_u^\pi))_u \in M^2([t, T])$.

As already noticed, the three state variables (X, S, η) in (2) and (5) form a Markov process, with infinitesimal generator given by

$$A^\pi V(t, x, s, k) := (L_x^\pi + L_{xs}^\pi) V(t, x, s, k) + \sum_{j=1}^m \lambda^{kj} \left[V(t, x, s, j) - V(t, x, s, k)\right] \tag{10}$$

for all $\pi \in \mathbb{R}^d$, $t \in [0, T]$, $x \in \mathbb{R}$, $s \in D$, $k = 1, \ldots, m$, where

$$\begin{aligned} L_x^\pi V &:= rxV_x + (\mu_k - r\mathbf{1}) \cdot \pi V_x + \frac{1}{2} \|\pi \Sigma_k\|^2 V_{xx}, \\ L_{xs}^\pi V &:= \mu_k \bar{s} V_s + \frac{1}{2} \mathrm{tr}\ (\bar{s} \Sigma_k \Sigma_k^T \bar{s} V_{ss}) + \pi \Sigma_k (\bar{s} \Sigma_k)^T V_{xs} \end{aligned}$$

with $\bar{s} := \text{diag } s$ for all $s \in D$, $g_k(t,s) = g(t,s,k)$ for $g = \mu$, σ, V_t, V_x and V_{xx} are the scalar derivatives with respect to the variables t and x, while V_s is the gradient with respect to the vector $s = (s^1, \ldots, s^d)$, V_{ss} is the Hessian matrix and V_{sx} is the gradient of V_x with respect to s. The operator A^π is linked to the process (X, S, η) via the so-called *Dynkyn formula*

$$\mathbb{E}[f(T, X_T^\pi, S_T, \eta_T)] - \mathbb{E}[f(t, X_t^\pi, S_t, \eta_t)] = \mathbb{E}\left[\int_t^T A^{\pi_u} f(u, , X_u^\pi, S_u, \eta_u)\, du\right] \tag{11}$$

for f regular enough.

We can now write the "integro-differential" Hamilton-Jacobi-Bellman (HJB) equation related to the utility maximization problem (9) above in a similar manner following the arguments of [19], and obtain

$$V_t^k + \sup_\pi A^\pi V = 0 \tag{12}$$

where $V^i(t,x,s) := V(t,x,s,i)$, $V(t,x,s) := (V(t,x,s,1), \ldots, V(t,x,s,m))$ and the "integrals" are on the space $\{1, \ldots, m\}$, and final condition

$$V^k(T,x,s) = U(x), \qquad (x,s,k) \in \mathbb{R} \times D \times \{1, \ldots, m\} \tag{13}$$

Solving for $\hat{\pi}^k$ which maximizes the Hamiltonian, we obtain

$$\hat{\pi}^k = -\frac{(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})V_x^k + \bar{s}V_{xs}^k}{V_{xx}^k}. \tag{14}$$

Notice that if V is independent of s, then the HJB equation simplifies to

$$V_t^k + \sup_h L_x^\pi V^k + \sum_{j=1}^m \lambda^{kj}\left[V^j - V^k\right] = 0 \tag{15}$$

and the optimal portfolio reduces to a Merton-type one

$$\hat{\pi}^k = -\frac{V_x^k}{V_{xx}^k}(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1}). \tag{16}$$

The rigorous link between the utility maximization problem (9) and the HJB equation (12) is given, as usual, by the following verification theorem. In order to present it, we follow the results in [11] and [13]. First define

$$\mathcal{D} := \left\{ \begin{array}{ll} f & = (f_1, \ldots, f_m) \in C^{1,2}([0,T] \times \mathbb{R} \times D; \mathbb{R}^m) \text{ such that} \\ & \forall t \in [0,T], \text{ the Dynkyn formula (11) holds } \forall \pi \in \Theta[t,T] \end{array} \right\}. \tag{17}$$

The usual choice for possibly discontinuous Markov processes is $\mathcal{D} := C_0^2([0,T] \times \mathbb{R})$, the C^2 functions vanishing at infinity: in fact for this space it is always possible to prove that the Dynkyn formula holds. However, this space is too small for our purposes, as typical utility functions are unbounded, so we have to define $\mathcal{D}$ more generally, as done also in [20].

We can now state the following verification theorem, which is a particular case of [13, Theorem III.8.1].

Theorem 2.2 (Verification Theorem). *Let $K \in \mathcal{D}$ be a classical solution to (12) with final condition (13), and assume that there exists an admissible control $\pi^* \in \Theta[t,T]$ such that*

$$\pi^*_u \in \arg\max_{\pi} A^\pi K(u, X^\pi_u, S_u, \eta_u) \qquad \mathbb{P}\text{-a.s. for all } u \in [t,T].$$

Then $K(t,x,s,k) = J^{\pi^*}(t,x,s,k) = V^k(t,x,s)$.

Thus, the utility maximisation problem boils down to finding a regular solution of the HJB equation. The usual procedure for this is to guess a particular solution for a given utility function U and to see whether this particular candidate satisfies the Verification Theorem above. Since the most challenging task in doing this is usually to check whether this candidate solution belongs to $\mathcal{D}$, here we present a technical lemma which will be used in the following sections.

Lemma 2.3. *If the Σ_k are bounded in (t,s), then $V \in \mathcal{D}$ if, for all $\bar{t} \in [0,T]$ and $\pi \in \Theta[\bar{t},T]$, we have that for all $k,j = 1,\ldots,m$,*

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|V_x(t,X_t,S_t,\eta_t)\pi_u\|^2 + \|V_s(t,X_t,S_t,\eta_t)\cdot S_u\|^2 \; dt\right] < +\infty, \tag{18}$$

$$\mathbb{E}\left[\int_{\bar{t}}^{T} |V(t,X_t,S_t,j) - V(t,X_t,S_t,k)|\lambda^{kj}(t,S_t) \; dt\right] < +\infty. \tag{19}$$

Proof. For $\bar{t} \in [0,T]$ and $\pi \in \Theta[\bar{t},T]$ we have

$$dV(t,X^\pi_t,S_t,\eta_t) = A^\pi V(t,X^\pi_t,S_t,\eta_{t-})\; dt + dM_t$$

where the process M is defined by $M_{\bar{t}} := 0$ and the dynamics

$$\begin{aligned} dM_t \;\; := \;\; & V_x\pi_t\Sigma_t \; dW_t + V_s \text{diag } S_t\Sigma_t \; dW_t + \\ & + \sum_{j,k=1}^{m} [V(t,X_t,S_t,j) - V(t,X_t,S_t,k)]\mathbf{1}_{\{k\}}(\eta_{t-})(dN^{kj}_t - \lambda^{kj}(t,S_t)\; dt) \end{aligned}$$

The lemma follows from the fact that the Dynkyn formula holds if M is a martingale. Sufficient conditions for this are Equation (19) and

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|V_x(t,X_t,S_t,\eta_t)\pi_u\Sigma(u,X_u,S_u,\eta_u)\|^2 + \|V_s(u,X_u,S_u,\eta_u)\text{diag } S_u\Sigma(u,X_u,S_u,\eta_u)\|^2 \; du\right] < +\infty$$

If the Σ_k are bounded in (t,s), then this is implied by Equation (18). □

Remark 2.4. The requirement that the Σ_k are bounded in (t,s) is quite natural: in fact, since the diffusion coefficient (conditioned to $\eta_{t-} = k$) in Equation (2) is diag $(S_t)\Sigma_k(t,S_t)$, a classical sufficient condition to have a unique solution is to require it (and the drift) to be Lipschitz and with sublinear growth with respect to S_t: this is morally equivalent to the fact that the Σ_k are bounded.

A first consequence of Theorem 2.2 is a generalization of the classical mutual fund theorem (see e.g. [5]). This result is obtained in the case when μ_k, Σ_k do not depend on S, and is valid with any utility function U such that Equation (15) has a smooth solution.

Corollary 2.5 (**Mutual fund theorem).** *If μ_k, Σ_k and λ^{kj} do not depend on S and Equation (15) has a smooth solution $V = (V^k)_k$, then the optimal portfolio strategy is given by the feedback control $\hat{\pi}_t := \hat{\pi}^k(t, X_t, S_t)|_{k=\eta_{t-}}$, where the functions $\hat{\pi}^k$, $k = 1, \dots, m$ are defined as in Equation (16).*

Proof. Since the μ_k, Σ_k and λ^{kj} do not depend on S, as well as the final condition $U(x)$, we can search for a solution of the form $V^k(t, x)$, thus $V_s^k = V_{ss}^k = V_{xs}^k = 0$ and the optimal strategy in Equation (14) becomes Equation (16). Substituting this in Equation (12), we obtain Equation (15): if this equation has a smooth solution, then the Verification Theorem 2.2 applies, and the theorem follows. □

Remark 2.6. Roughly speaking, results known as "mutual fund theorems" (for a much more general tractation see [21]) say that the optimal portfolio consists of a possibly dynamic allocation between two fixed mutual funds: in this particular situation, the first fund consists only of the risk-free asset B, while the second fund is given by the fixed vector

$$(\Sigma_{\eta_{t-}}(t)\Sigma_{\eta_{t-}}^T(t))^{-1}(\mu_{\eta_{t-}}(t) - r\mathbf{1})$$

for all $t \in [0, T]$, which *does not depend on the particular utility function used nor on the individual prices of the risky assets*, but still depends on time t and on the state η_{t-}. The amount of this fund to be taken is given by the scalar

$$-\frac{V_x^{\eta_{t-}}(t, X_t)}{V_{xx}^{\eta_{t-}}(t, X_t)}$$

which is typically positive as the functions V^k are typically increasing and concave.

Of course, the major assumption of Theorem 2.2 and Corollary 2.5 is to have a smooth solution V. This is satisfied in the following three particular cases, which are quite standard, namely the exponential, logarithmic and power utility functions: what really happens is that in some of these cases one obtains results also with more general assumptions.

3 Exponential utility

We now analyze the particular case when $U(x) = -\alpha e^{-\alpha x}$, with $\alpha > 0$.

Lemma 3.1. *Assume that $U(x) = -\alpha e^{-\alpha x}$, with $\alpha > 0$, and that for all $k = 1, \dots, m$ the functions μ_k, $\Sigma_k\Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all (t, s), $\Sigma_k^{-1}\mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0, T] \times D)$ for all $k, j = 1, \dots, m$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\dots,m} \in C_b^{1,2}([0, T] \times D; \mathbb{R}^m)$ for the following system of PDEs:*

$$\begin{cases} C_t^k + rs \cdot C_s^k + \dfrac{1}{2}\mathrm{tr}\left(\bar{s}\Sigma_k\Sigma_k^T\bar{s}C_{ss}^k\right) - \dfrac{e^{-r(T-t)}}{\alpha}\left[\displaystyle\sum_{j=1}^m \left(e^{-\alpha\phi(C^k - C^j)} - 1\right)\lambda^{kj} + \dfrac{1}{2}z_k^2\right] = rC^k, \\ C^k(T) = 0 \end{cases} \tag{20}$$

where the functions ϕ and z_k^2 are defined as

$$\phi(t) := e^{r(T-t)}, \quad z_k^2(t, s) := (\mu_k(t, s) - r\mathbf{1})^T(\Sigma_k(t, s)\Sigma_k^T(t, s))^{-1}(\mu_k(t, s) - r\mathbf{1}) \tag{21}$$

2. The function

$$V^k(t,x,s) = -\alpha e^{-\alpha\phi(t)(x-C^k(t,s))} \tag{22}$$

with $\phi(t) := e^{r(T-t)}$*, belongs to* $\mathcal{D}$ *and satisfies the HJB equation (15).*

3. The optimal portfolio strategy $\hat{\pi}^k_t$ *is given by*

$$\hat{\pi}_t := \left.\frac{(\Sigma_k(t,S_t)\Sigma_k(t,S_t)^T)^{-1}(\mu_k(t,S_t) - r\mathbf{1}) + \alpha\phi(t)\mathrm{diag}\,(S_t)C^k_s(t,S_t)}{\alpha\phi(t)}\right|_{k=\eta_{t-}} \tag{23}$$

Proof. Point 1. follows from [4, Theorem 2.4]. For point 2., the partial derivatives of V^k are given by

$$\begin{cases} V^k_t &= \left[C^k_t + rx - rC^k\right]\alpha\phi V^k \\ V^k_x &= -\alpha\phi V^k \\ V^k_{xx} &= (\alpha\phi)^2 V^k \\ V^k_s &= C^k_s \alpha\phi V^k \\ V^k_{ss} &= [\alpha\phi C^k_s \otimes C^k_s + C^k_{ss}]\alpha\phi V^k \\ V^k_{xs} &= -C^k_s(\alpha\phi)^2 V^k. \end{cases}$$

In order to prove $V \in \mathcal{D}$, we check Equations (18–19): firstly,

$$\begin{aligned} &\mathbb{E}\left[\int_{\bar{t}}^T \|V_x(t,X_t,S_t,\eta_t)\pi_u\|^2 + \|V_s(t,X_t,S_t,\eta_t)\cdot S_u\|^2\,dt\right] = \\ &= \mathbb{E}\left[\int_{\bar{t}}^T \|\alpha\phi(t)U(X_t)e^{\alpha\phi(t)C(t,S_t)}\pi_t\|^2\,dt\right] + \mathbb{E}\left[\int_{\bar{t}}^T \|\alpha\phi(t)V(t,X_t,S_t,\eta_t)C_s(t,X_t,S_t,\eta_t)\cdot S_t\|^2\,dt\right] \end{aligned}$$

Since C and ϕ are bounded, the first addend is finite since $\pi \in \Theta[\bar{t},T]$, and by the same argument the second addend reduces to

$$M\mathbb{E}\left[\int_{\bar{t}}^T \|e^{-\alpha\phi(t)X^\pi_t}S_t\|^2\,dt\right] \le M(T-\bar{t})\mathbb{E}\left[\int_{\bar{t}}^T e^{-2\alpha\phi(t)X^\pi_t}\,dt\right]^{1/2}\mathbb{E}\left[\int_{\bar{t}}^T \|S_t\|^2\,dt\right]^{1/2}$$

for a suitable M: the final product is finite by standard SDE estimates (see for example [13, Appendix D], so Equation (18) is satisfied. As concerns Equation (19), it reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^T e^{-\alpha\phi(t)X^\pi_t}|e^{\alpha\phi(t)C^j(t,S_t)} - e^{\alpha\phi(t)C^k(t,S_t)}|\lambda^{kj}(t,S_t)\,dt\right] \le M\mathbb{E}\left[\int_{\bar{t}}^T e^{-\alpha\phi(t)X^\pi_t}\,dt\right]$$

for a suitable M: as before, this quantity is finite, so also Equation (19) is satisfied. Thus, $V \in \mathcal{D}$ by Lemma 2.3. Substituting its derivatives in the operators L^π_x and L^π_{xs} appearing in Equation (12), we get

$$\begin{aligned} L^\pi_x V^k &:= \alpha\phi V^k\left[-rx - (\mu_k - r\mathbf{1})\cdot\pi + \frac{1}{2}\alpha\phi\|\pi\Sigma_k\|^2\right], \\ L^\pi_{xs} V^k &:= \alpha\phi V^k\left[\mu_k\bar{s}C^k_s + \frac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}[\alpha\phi C^k_s\otimes C^k_s + C^k_{ss}]) - \alpha\phi\pi\Sigma_k(\bar{s}\Sigma_k)^T C^k_s\right] \\ &:= \alpha\phi V^k\left[\mu_k\bar{s}C^k_s + \frac{1}{2}\alpha\phi\|\bar{s}C^k_s\Sigma_k\|^2 + \frac{1}{2}\mathrm{tr}\,(\bar{s}\Sigma_k\Sigma_k^T\bar{s}C^k_{ss}) - \alpha\phi\pi\Sigma_k(\bar{s}\Sigma_k)^T C^k_s\right] \end{aligned}$$

The maximizer in Equation (14) becomes

$$\hat{\pi}^k = \frac{(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1}) + \alpha\phi\bar{s}C_s^k}{\alpha\phi}$$

Plugging all into Equation (12) and dividing by $\alpha\phi V^k$, we obtain Equation (20), which is satisfied by C. Finally, point 3. follows from the $\hat{\pi}^k$ just obtained. □

Remark 3.2. Point 2. is not a surprise here: in fact, in the exponential case one can check right from the definition of V in Equation (9) and from the dynamics of X in Equation (5) that

$$V(t, x, s, \eta) = e^{-\alpha\phi(t)x} V(t, 0, s, \eta)$$

for all (t, x, s, η), so that the value function V is (exp-)affine in x.

As a particular case, we can see that the mutual fund Theorem 2.5 holds true in this situation.

Corollary 3.3. *If λ^{kj} and z_k^2 do not depend on s for all $k, j = 1, \ldots, m$, then the discounted wealth invested in the risky assets is given by*

$$e^{r(T-t)}\hat{\pi}_t = \left.\frac{(\Sigma_k(t, S_t)\Sigma_k(t, S_t)^T)^{-1}(\mu_k(t, S_t) - r\mathbf{1})}{\alpha}\right|_{k=\eta_{t-}} \tag{24}$$

Moreover, if also μ_k and Σ_k do not depend on s for all $k = 1, \ldots, m$, then the above optimal discounted wealth invested in the risky assets only depends on (t, η_{t-}).

Proof. In this case, the solution of the system of ODEs

$$\begin{cases} C_t^k - \dfrac{e^{-r(T-t)}}{\alpha}\left[\displaystyle\sum_{j=1}^m \left(e^{-\alpha\phi(C^k - C^j)} - 1\right)\lambda^{kj} + \frac{1}{2}z_k^2\right] = rC^k, \\ C^k(T) = 0 \end{cases}$$

is also solution of Equation (20), so that $C_s \equiv 0$. Thus,

$$\hat{\pi}_t = \left.\frac{(\Sigma_k(t, S_t)\Sigma_k(t, S_t)^T)^{-1}(\mu_k(t, S_t) - r\mathbf{1})}{\alpha\phi(t)}\right|_{k=\eta_{t-}}$$

and by multiplying for $\phi(t)$ we obtain the desired result. □

4 Logarithmic utility

In this and the following section, as we will have the constraint $X_t > 0$ for all $t \in [0, T]$, as definition of strategy we adopt h (the proportion of wealth in the risky assets) instead of π. This means that in the infinitesimal generator Equation (10) we must substitute $L_x^{xh} + L_{xs}^{xh}$ to $L_x^\pi + L_{xs}^\pi$.

In the case of a logarithmic utility function, the optimal portfolio is of the general form in Equation (14) even when μ_k, Σ_k and λ^{kj} depend on S.

Proposition 4.1. *Assume that $U(x) = \log x$, and that for all $k = 1, \ldots, m$ the functions μ_k, $\Sigma_k \Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all (t, s), $\Sigma_k^{-1} \mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0,T] \times D)$ for all $k, j = 1, \ldots, m$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\ldots,m} \in C_b^{1,2}([0,T] \times D; \mathbb{R}^m)$ for the following system of PDEs:*

$$\begin{cases} C_t^k + r + \mu_k \bar{s} C_s^k + \dfrac{1}{2} \mathrm{tr}\, (\bar{s} \Sigma_k \Sigma_k^T \bar{s} C_{ss}^k) + \dfrac{1}{2} z_k^2 + \displaystyle\sum_{j=1}^{m} \left(C^j - C^k \right) \lambda^{kj} = 0, \\ C^k(T) = 0. \end{cases} \tag{25}$$

where the functions ϕ and z_k^2 are defined as in Equation (21).

2. *The function*

$$V^k(t, x, s) = \log x + C^k(t, s) \tag{26}$$

belongs to $\mathcal{D}$ and satisfies the HJB equation (15).

3. *The optimal portfolio proportion $\hat{h}_t^k$ is given by*

$$\hat{h}_t := (\Sigma_k(t, S_t) \Sigma_k(t, S_t)^T)^{-1} (\mu_k(t, S_t) - r\mathbf{1})\big|_{k=\eta_{t-}} \tag{27}$$

Proof. Analogously as before, point 1. follows from [4, Theorem 2.4]. For point 2., the partial derivatives of V_k are now given by

$$V_t^k = C_t^k, \qquad V_x^k = \frac{1}{x}, \qquad V_{xx}^k = -\frac{1}{x^2}, \qquad V_s^k = C_s^k, \qquad V_{ss}^k = C_{ss}^k, \qquad V_{sx}^k = 0,$$

In order to prove $V \in \mathcal{D}$, we check Equations (18–19): firstly, Equation (18) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|h_t\|^2 \, dt \right] + \mathbb{E}\left[\int_{\bar{t}}^{T} \|C_s(t, S_t, \eta_t) \cdot S_t\|^2 \, dt \right]$$

Since C is bounded, the second term is finite, and the first term is finite by definition of $\Theta[\bar{t}, T]$. Equation (19) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^{T} |C(t, S_t, j) - C(t, S_t, k)| \lambda^{kj}(t, S_t) \, dt \right] < +\infty$$

which is true, due to the boundedness of C and of the λ^{kj}. Thus, $V \in \mathcal{D}$. Substituting its derivatives in the operators L_x^{xh} and L_{xs}^{xh} appearing in Equation (12), we get

$$\begin{aligned} L_x^{xh} V^k &:= r + (\mu_k - r\mathbf{1}) \cdot h - \frac{1}{2} \|h \Sigma_k\|^2, \\ L_{xs}^{xh} V^k &:= \mu_k \bar{s} C_s^k + \frac{1}{2} \mathrm{tr}\, (\bar{s} \Sigma_k \Sigma_k^T \bar{s} C_{ss}^k) \end{aligned}$$

The maximizer in Equation (14) becomes

$$\hat{h}^k = (\Sigma_k \Sigma_k^T)^{-1} (\mu_k - r\mathbf{1})$$

Plugging all into Equation (12), we obtain Equation (25), which is satisfied by C. Finally, point 3. follows from the $\hat{h}^k$ just obtained. □

This result in some sense extends the result of Merton [18], in the sense that the optimal portfolio has the form of the Merton optimal portfolio, even if the coefficients μ_k and Σ_k can depend on the price of the risky assets S in general. If they do not depend on t and S, however, we retrieve the usual "constant proportions" result, which follows.

Corollary 4.2. *If μ_k and Σ_k do not depend on (t,s) for all $k=1,\dots,m$, then the optimal portfolio proportions in the risky assets are given by*

$$\hat{h}_t := (\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})\big|_{k=\eta_{t-}}$$

which only depends on η_{t-}.

Proof. The proof is straightforward from point 3. of Proposition 4.1. □

5 Power utility

In the case of a power utility function, we do not get general results as in the two previous cases, unless we assume that μ_k, Σ_k and λ^{kj} do not depend on s.

Proposition 5.1. *Assume that $U(x) = x^\gamma/\gamma$, with $\gamma < 1$, $\gamma \neq 0$, and that for all $k,j = 1,\dots,m$ the functions μ_k, Σ_k and λ^{kj} do not depend on s; besides, μ_k, $\Sigma_k\Sigma_k^T$ are locally Lipschitz and bounded, Σ_k is nonsingular for all t, $\Sigma_k^{-1}\mu_k$ is bounded and $\lambda^{kj} \in C_b^1([0,T])$. Then:*

1. *There exists a unique classical solution $C := (C^k)_{k=1,\dots,m} \in C_b^{1,2}([0,T];\mathbb{R}^m)$ for the following system of ODEs:*

$$\begin{cases} C_t^k + r + \dfrac{1}{2}\dfrac{1}{(1-\gamma)} z_k^2 + \dfrac{1}{\gamma}\displaystyle\sum_{j=1}^m \left(e^{\gamma(C^j - C^k)} - 1\right)\lambda^{kj} &= 0, \\ C^k(T) &= 0. \end{cases} \tag{28}$$

   where the functions ϕ and z_k^2 are defined as in Equation (21).

2. *The function*

$$V^k(t,x,s) = \frac{\left(xe^{C^k(t)}\right)^\gamma}{\gamma} \tag{29}$$

   belongs to $\mathcal{D}$ and satisfies the HJB equation (15).

3. *The optimal portfolio proportion $\hat{h}_t^k$ is given by*

$$\hat{h}_t := \frac{1}{1-\gamma}(\Sigma_k(t)\Sigma_k(t)^T)^{-1}(\mu_k(t) - r\mathbf{1})\Big|_{k=\eta_{t-}} \tag{30}$$

Proof. Analogously as before, point 1. follows from [4, Theorem 2.4]. For point 2., the partial derivatives of V_k are now given by

$$V_t^k = \gamma C_t^k V^k, \qquad V_x^k = \frac{\gamma}{x} V^k, \qquad V_{xx}^k = -\frac{\gamma(1-\gamma)}{x^2} V^k, \qquad V_s^k = V_{sx}^k = V_{ss}^k = 0.$$

In order to prove $V \in \mathcal{D}$, we check Equations (18–19): firstly, Equation (18) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \|\gamma V(t, X_t, S_t, \eta_t) h_t\|^2 \, dt\right]$$

which is finite by definition of $\Theta[\bar{t}, T]$. Equation (19) reduces to

$$\mathbb{E}\left[\int_{\bar{t}}^{T} \frac{1}{\gamma} X_t^{\gamma} |e^{\gamma C^j(t)} - e^{\gamma C^k(t)}| \lambda^{kj}(t) \, dt\right] \leq M \mathbb{E}\left[\int_{\bar{t}}^{T} X_t^{\gamma} \, dt\right]$$

for a suitable M, since C and the λ^{kj} are bounded. This quantity is finite by definition of $\Theta[\bar{t}, T]$. Thus, $V \in \mathcal{D}$. Substituting its derivatives in the operators L_x^{xh} and L_{xs}^{xh} appearing in Equation (12), we get

$$\begin{aligned} L_x^{xh} V^k &:= \gamma V^k \left(r + (\mu_k - r\mathbf{1}) \cdot h - \frac{1}{2}(1-\gamma)\|h\Sigma_k\|^2\right), \\ L_{xs}^{xh} V^k &:= 0 \end{aligned}$$

The maximizer in Equation (14) becomes

$$\hat{h}^k = \frac{(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})}{(1-\gamma)}.$$

Plugging all into Equation (15) and dividing by γV^k, we obtain Equation (28), which is satisfied by C. Finally, point 3. follows from the $\hat{h}^k$ just obtained. □

Also this result in some sense extends the result of Merton [18], in the sense that the optimal portfolio has the form of the Merton optimal portfolio. If μ_k and Σ_k do not depend on t, we again retrieve the usual "constant proportions" result, which follows.

Corollary 5.2. *If μ_k and Σ_k do not depend on (t, s) for all $k = 1, \ldots, m$, then the optimal portfolio proportions in the risky assets are given by*

$$\hat{h}_t := \frac{1}{1-\gamma} (\Sigma_k \Sigma_k^T)^{-1} (\mu_k - r\mathbf{1}) \Big|_{k=\eta_{t-}}$$

which only depends on η_{t-}.

Proof. The proof is straightforward from point 3. of Proposition 4.1. □

Remark 5.3. If μ_k, Σ_k or λ^{kj} depend also on s, then Equation (28) must be modified as follows:

$$\begin{cases} C_t^k + r + \dfrac{1}{2}\dfrac{1}{(1-\gamma)} z_k^2 + \dfrac{1}{\gamma}\displaystyle\sum_{j=1}^{m} \left(e^{\gamma(C^j - C^k)} - 1\right) \lambda^{kj} + \\ \qquad + \left(\mu_k - \dfrac{1}{1-\gamma}(\mu_k - r\mathbf{1})\right) \bar{s} C_s + \dfrac{1}{2} \mathrm{tr}\,(\bar{s}\Sigma_k \Sigma_k^T \bar{s} C_{ss}) + \dfrac{1}{2}\dfrac{\gamma}{1-\gamma} \|\bar{s} C_s \Sigma_k\|^2 = 0, \\ C^k(T) = 0. \end{cases} \tag{31}$$

i.e. the additional terms in the second line must be included, one of which is a nonlinear function of the gradient C_s. This is a quasilinear system of PDEs, to which the results of [4] do not apply, and that in general needs a theory which is more complex and beyond the scopes of this paper.

We are now able to reconsider the initial Example 1.1 and to make it rigorous.

Example 5.4 (Example 1.1 continued). *Assume $d = 2$ and let $\mu_k - r\mathbf{1} \equiv (0.01, 0.01)$ for $k = 1, 2$ and*

$$\Sigma_k := \begin{cases} \begin{pmatrix} 0.04 & 0.004 \\ 0.004 & 0.04 \end{pmatrix} & \text{for } k = 1, \\ \\ \begin{pmatrix} 0.09 & 0.054 \\ 0.054 & 0.09 \end{pmatrix} & \text{for } k = 2, \end{cases}$$

Then the optimal portfolio strategy for an investor with $U(x) = x^\gamma/\gamma$ with $\gamma = 0.1$, by Equation (33) is given by

$$\hat{h}_t := \frac{1}{1-\gamma}(\Sigma_k \Sigma_k^T)^{-1}(\mu_k - r\mathbf{1})\Big|_{k=\eta_{t-}} = (0.2525, 0.2525)\mathbf{1}_{\{\eta_{t-}=1\}} + (0.0771, 0.0771)\mathbf{1}_{\{\eta_{t-}=2\}}$$

Thus, our optimal investor always switches between holding 25.25% of its wealth in each of the two risky assets in the "normal" state, and 7.71% in each of the two risky assets during a "crisis" state.

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