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Explicit solutions for shortfall risk minimization in multinomial models

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1. Introduction

In this paper we show how to obtain explicit solutions for the problem of shortfall risk minimization in significant multinomial models with one or several risky assets. First we solve the problem when the market is complete, finding both the minimall shortfall risk as well as the optimal strategy. Then we indicate how the situation can change in incomplete markets, by solving (under some technical assumptions) the problem in the simplest case of an incomplete market, that is the case of a single risky asset driven by a trinomial model.

In incomplete markets there are several possible criteria for hedging a risky position. The safest one is the superhedging criterion, that allows one to eliminate the risk completely, but requires in general too much initial capital. One may then ask by how much is it possible to lower the initial capital if one is willing to accept some risk or,

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dually, what is the risk corresponding to an initial capital less than what is required for superhedging.

The shortfall risk minimization approach allows one to deal with these issues. Given a market with a riskless asset and K risky assets, let H be a liability to be hedged at some fixed future time N. Denote by V_N^φ the value at time N of a portfolio corresponding to a self-financing investment strategy $\varphi \in \mathcal{A}$ (where this set can describe some additional constraints such as no short selling, no borrowing, and similar). The problem is to find the so-called *minimal shortfall risk* $J(0, S_0, V_0)$, where $J(n, \cdot, \cdot)$ is defined, for $n = 0, \dots, N$, as

$$J(n, S_n, V_n) := \inf_{\varphi \in \mathcal{A}} E^{\mathcal{P}}_{S_n, V_n} \left\{ [H(S_N) - V_N^\varphi]^+ \right\} \tag{1}$$

for given initial values $S_n = (S_n^1, \dots, S_n^K)$ of the assets in the portfolio and initial capital V_n. Problems of this type have recently attracted considerable attention (see e.g. Cvitanić and Karatzas (1999) for a bibliography). The *superreplication capital* for the claim H can be described in terms of J as

$$V_0^*(S_0) := \inf\{V_0 \mid J(0, S_0, V_0) = 0\}. \tag{2}$$

In fact, it follows by definition of J that if $V_0 > V_0^*(S_0)$ then there exists at least one strategy φ^* such that $V_N^{\varphi^*} \geq H$ a.s.

In the present paper we base ourselves on Runggaldier *et al.* (2001), where the authors, by imposing the self-financing requirement as the only constraint on the investment strategies, give a general description of this approach and apply it, in particular, to a binomial model for the risky assets. In this paper we do the same with multinomial models. First we analyse the case when the market is complete: we can obtain this situation if for example we decide to operate in a homogeneous section of a market (as may be a bond market where all the assets are driven by a single factor, which is typically the short rate), or if one of the risky assets is an underlying primary asset and the others are derivatives of it. Then we analyse the simplest case of an incomplete market, that is when we have only one risky asset which is governed by a trinomial model. In both these situations, we obtain analytic solutions for the minimum value of the shortfall as well as for the optimal hedging strategy; these solutions turn out to be a generalisation of those in Runggaldier *et al.* (2001). The key issue here (as in the paper cited) in obtaining analytic formulae is that the value function (1) preserves the same form at each time step n.

The paper is organised as follows. In Section 2 we present the general multinomial model and formulate the shortfall risk minimization problem. In Section 3 we characterise a minimising admissible strategy and we compute the minimal shortfall risk in the case when the market is complete. In Section 4 we compute a minimising admissible strategy and the minimal shortfall risk in the case when the risky asset is governed by a trinomial tree.

2. The model

We consider a discrete time market model with the set of dates $0, \dots, N$ and with $K+1$ primary traded securities: a riskless investment S^0 and K risky assets S^i, $i = 1, \dots, K$. We assume that the prices of the risky assets satisfy

$$S^i_{n+1} = S^i_n \omega^i_n, \quad i = 1, \dots, K, \tag{3}$$

where, for each $i = 1, \dots, K$, $\{\omega^i_n\}_{n=0,\dots,N-1}$ is a sequence of i.i.d. random variables taking only a finite number M of real values $(a^j_i)_{j=1,\dots,M}$. For the sake of simplicity, we define the probability space as the minimal one for our model, by letting $\Omega = \{a^1, \dots, a^M\}^N$ (we use the notation $a^j = (a^j_1, \dots, a^j_K)$, $j = 1, \dots, M$), the σ-algebra $\mathcal{F}$ be composed by all the subsets of Ω, and the probability law be defined by

$$p_j := \mathcal{P}\{\omega_n = a^j\}, \quad j = 1, \dots, M,$$

where $\omega_n = (\omega^1_n, \dots, \omega^K_n)$. We are interested in the evaluation of a future position at time N without intermediate payments, so we can assume without loss of generality $S^0_n := 1$ by letting S^i be the discounted prices of the i-th risky asset, $i = 1, \dots, K$. With this convention, it is sufficient to know only the prices S^i, $i = 1, \dots, K$, of the risky assets. We thus let $S_n = (S^1_n, \dots, S^K_n)$.

We denote by $\varphi = \big((\eta_n, \psi^1_n, \dots, \psi^K_n)\big)_{n=0,\dots,N-1}$ a *portfolio strategy*, where η_n stands for the amount invested in the riskless asset and ψ^i_n, $1 \le i \le K$, stands for the number of units of the i-th asset that are held in the portfolio in period n. The value of the portfolio V^φ is then defined by

$$V^\varphi_n := \eta_n + \langle \psi_n, S_n \rangle, \qquad n = 0, \dots, N,$$

where $V^\varphi_0 = V_0$ is a given real number, corresponding to the initial capital, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^K$. We assume that φ is adapted to the filtration $\{\mathcal{F}_n\}_{n=0,\dots,N}$ generated by S, and satisfies the self-financing property:

$$\eta_n + \langle \psi_n, S_{n+1} \rangle = \eta_{n+1} + \langle \psi_{n+1}, S_{n+1} \rangle, \quad n = 0, \dots, N-1.$$

If the portfolio is self-financing, then the process ψ is sufficient to characterise it, since $\eta_n = V_n^\varphi - \langle \psi_n, S_n \rangle$. Using the notation $S_{n+1} = \big(S_n; a^j\big)$ for $S_{n+1}^i = S_n^i a_i^j$, $i = 1, \dots, K$, and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^K$, the self-financing property can be rewritten as

$$V_{n+1}^\varphi = V_n^\varphi + \langle \psi_n, (S_n; \omega_n - \mathbf{1}) \rangle.$$

Consider a European contingent claim $H(S_N)$. As mentioned in the introduction, we are interested in finding the *minimal shortfall risk* $J(0, S_0, V_0)$ for given initial prices of the risky assets S_0 and given initial capital $V_0 < V_0^*(S_0)$, where J is defined by Equation (1), V^* is defined in Equation (2) and $\mathcal{A}$ denotes the set of all adapted and self-financing strategies.

3. Shortfall risk minimization in a complete market model

Now we consider the case when in our previous model $M = K+1$, and the market is complete (see Florio and Runggaldier (1999) for some motivations and a possible construction of such a model). In complete markets the perfect-hedging criterion allows one to eliminate the risk completely and there exists a unique equivalent martingale measure $\mathcal{Q}$. The random variables $(\omega_n)_n$ are i.i.d. under $\mathcal{Q}$, which is characterised by a vector $q = (q_1, \dots, q_M)$, where

$$q_j = \mathcal{Q}\{\omega_n = a^j\}, \quad j = 1, \dots, M, \quad n = 0, \dots, N,$$

such that the vector q is the unique solution of the equations

$$\langle q, a_i \rangle = 1, \quad i = 1, \dots, K, \tag{4}$$

$$\langle q, \mathbf{1} \rangle = 1, \tag{5}$$

where $a_i = (a_i^1, \dots, a_i^M)$ (recall that, by assuming $S_n^0 := 1$ we implicitly assume that the riskless interest rate is equal to zero). Moreover, the superreplication capital $V_n^*(S_n)$ is equal to the expectation of $H(S_N)$ under $\mathcal{Q}$:

$$V_n^*(S_n) = E^{\mathcal{Q}}\big\{H(S_N) | \mathcal{F}_n\big\}, \quad n = 0, \dots, N.$$

In order to solve the problem (1) in the present situation, we notice that if we subtract Equation (5) from each of the K equations in (4), we obtain the conditions

$$\langle q, a_i - \mathbf{1} \rangle = 0, \quad i = 1, \dots, K, \tag{6}$$

which together with (5) give another characterisation of the martingale measure. This characterisation allows one to obtain a proof of the following theorem.

Theorem 1. *Consider a European contingent claim $H_N = H(S_N)$. Then*

$$J\big(n, S_n, V_n\big) = \left(\frac{p_r}{q_r}\right)^{N-n} \big[V_n^*(S_n) - V_n\big]^+ \tag{7}$$

for all $n = N-1, \dots, 0$, where r is the index for which the quantity p_j/q_j, $j = 1, \dots, M$, assumes its minimal value. Moreover, the optimal strategy ψ_n at time n, $n = 0, \dots, N-1$, is the solution of the system of K linear equations

$$V_{n+1}^*((S_n; a^j)) - V_n - \langle \psi_n, (S_n; a^j - \mathbf{1})\rangle = 0, \tag{8}$$

$j = 1, \dots, M$, $j \neq r$, which gives a portfolio V_N^φ such that

$$H(S_N) - V_N^\varphi = \left(\frac{1}{q_r}\right)^N [V_0^*(S_0) - V_0] \mathbf{1}_{\{\omega_n = a^r \;\; \forall n = 0, \dots, N-1\}}. \tag{9}$$

Proof. The proof uses the classical Dynamic Programming algorithm (see e.g. Bertsekas (1976)). To this end, we define the quantities

$$j(n, s, v, \psi) = E\{J(n, S_n, V_n^\psi) \mid S_{n-1} = s, V_{n-1} = v\}. \tag{10}$$

We start from $n = N-1$ and we obtain

$$J\Big(N-1, S_{N-1}, V_{N-1}\Big) = \inf_\psi j(N-1, S_{N-1}, V_{N-1}, \psi).$$

The function to be minimized in ψ_{N-1} is a linear combination of the M piecewise affine functions

$$\psi \to \big[H\big((S_{N-1}; a^j)\big) - V_{N-1} - \langle \psi, (S_{N-1}; a^j - \mathbf{1})\rangle\big]^+, \tag{11}$$

$j = 1, \dots, M$. The epigraph of the function to be minimized is a convex set with a finite number of extremal points: therefore the infimum is achieved at one of these points. These extremal points are obtained by putting equal to zero $M-1$ of the M functions in Equation (11). Thus the k-th extremal point is obtained by imposing the linear system (8) for $n = N-1$, $j = 1, \dots, M$, $j \neq k$, in the unknown quantity $\psi^{(k)}$ (notice that $V^*(N, \cdot) = H(\cdot)$). For $\psi_{N-1} = \psi^{(k)}$ we obtain

$$\begin{aligned} &j(N-1, S_{N-1}, V_{N-1}, \psi^{(k)}) = \\ &= p_k \Big[H\Big((S_{N-1}; a^k)\Big) - V_{N-1} - \langle \psi^{(k)}, (S_{N-1}; a^k - \mathbf{1})\rangle\Big]^+. \end{aligned}$$

If $V_{N-1} < V^*(N-1, S_{N-1})$, then the quantity above must be positive, otherwise $J\big(N-1, S_{N-1}, V_{N-1}\big) = 0$. By summing the left hand sides of Equations (8), we obtain

$$\begin{aligned}
&\frac{p_k}{q_k} \cdot q_k \Big[H\big((S_{N-1}; a^k)\big) - V_{N-1} - \langle \psi^{(k)}, (S_{N-1}; a^k - \mathbf{1})\rangle\Big]^+ = \\
&= \frac{p_k}{q_k} \sum_{j=1}^{M} q_j \Big[H\big((S_{N-1}; a^j)\big) - V_{N-1} - \langle \psi^{(k)}, (S_{N-1}; a^j - \mathbf{1})\rangle\Big]^+ = \\
&= \frac{p_k}{q_k} \Big(E^{\mathcal{Q}}\Big\{H(S_N)|\mathcal{F}_{N-1}\Big\} - V_{N-1}\Big)^+,
\end{aligned}$$

where the last equality follows by reordering terms of the sum and the scalar product and using Equation (6). Taking the minimum index r over $k = 1, \dots, M$ we obtain Equation (7) for $n = N-1$, and also that the minimizing strategy is given by the solution $\psi^{(r)}$ of the linear system of the equations in (8) for $j = 1, \dots, M$, $j \neq r$.

We now proceed by induction with respect to n. Equations (7) and (8) can be easily derived by using the Dynamic Programming algorithm with the same arguments as in the first step (the missing details are similar to those in Runggaldier *et al.* (2001)). Finally, in order to prove Equation (9) one can proceed as in Favero (2001). □

Remark 1. The previous theorem gives a generalization of the results in Runggaldier *et al.* (2001): in fact, the minimum shortfall turns out to be the difference between the arbitrage free price and the initial capital, times the minimum $(p_r/q_r)^N$ of the density $d\mathcal{P}/d\mathcal{Q}$, and the optimal strategy turns out to be such that the only state of nature in which there is a positive shortfall is the one in which the marginal density p_r/q_r is minimal. In particular, letting $K = 1$, we obtain exactly the same results as in Runggaldier *et al.* (2001).

4. Shortfall risk minimization in a trinomial model

Now we consider the case when $K = 1$ and $M = 3$. We assume that $\{\omega_n\}_{n=0,\dots,N-1}$ is a sequence of real i.i.d. random variables taking only three real values u, m, d satisfying $0 < d < m < u$ and $d < 1 < u$, with probability law

$$p_1 := \mathcal{P}\{\omega_n = u\}, \quad p_2 := \mathcal{P}\{\omega_n = m\}, \quad p_3 := \mathcal{P}\{\omega_n = d\}.$$

The market is not complete, so there are infinitely many martingale measures. Unlike the complete case, here the marginals of each ω_n, $n = 0, \dots, N-1$ under a generic martingale measure depend on the past (here represented by $\mathcal{F}_n$), possibly in a path-dependent way. If for a generic marginal of an equivalent martingale measure $\mathcal{Q}$ we put (by omitting for ease of notation the dependence on $\mathcal{F}_n$)

$$q_1 = \mathcal{Q}\{\omega_n = u\}, \quad q_2 = \mathcal{Q}\{\omega_n = m\}, \quad q_3 = \mathcal{Q}\{\omega_n = d\},$$

then q_1, q_2, q_3 must satisfy (see Equations (4) and (5))

$$\begin{cases} q_1 + q_2 + q_3 = 1, \\ q_1 u + q_2 m + q_3 d = 1, \\ q_1, q_2, q_3 > 0. \end{cases} \tag{12}$$

The solutions lie in a 1-dimensional affine space, thus we can describe any marginal law of the $(\omega_n)_n$ under an equivalent martingale measure as a convex combination of the two extremal measures. These extremal measures change according to whether $m \geq 1$ or $m \leq 1$: the first one is Q_0, characterised by

$$(q_1^0, q_2^0, q_3^0) = \begin{cases} \left(0, \dfrac{1-d}{m-d}, \dfrac{m-1}{m-d}\right) & \text{if } m \geq 1, \text{ and} \\[2ex] \left(\dfrac{1-m}{u-m}, \dfrac{u-1}{u-m}, 0\right) & \text{if } m \leq 1. \end{cases}$$

The second is Q_1, characterised by

$$(q_1^1, q_2^1, q_3^1) = \left(\frac{1-d}{u-d}, 0, \frac{u-1}{u-d}\right).$$

The measures Q_0 and Q_1 are absolutely continous with respect to the marginal laws of the $(\omega_n)_n$ under $\mathcal{P}$, but not equivalent. The set of all the marginals of a generic equivalent martingale measure is thus described as

$$Q_t = tQ_1 + (1-t)Q_0, \quad 0 < t < 1. \tag{13}$$

One can obtain the generic equivalent martingale measure by pasting together various single period marginals, each one corresponding to a non-terminal node of the tree of all the possible evolutions of the risky asset S_n (see Pliska (1997) for details); since each marginal depends on a parameter between 0 and 1, it is possible to describe the generic equivalent martingale measure $\mathcal{Q}_t$ via a k-dimensional parameter $t = (t_1, \dots, t_k)$, $0 < t_i < 1$, $k = 3^{N-1}$. If one relaxes the constraints to $0 \leq t_i \leq 1$, $k = 3^{N-1}$, one obtains a so-called *linear pricing measure* (see Pliska (1997)). Under such a measure (which in general is not equivalent to $\mathcal{P}$), S is a martingale, as is V^{φ} for any self-financing strategy φ.

We finally notice that the solution of the two equations in (12) that we obtain by imposing that the third parameter (q_3 if $m \geq 1$, q_1 if $m \leq 1$) be zero gives a measure Q^* which is not the marginal of a probability measure, because we have

$$q_1^* = \frac{1-m}{u-m} < 0, \quad q_2^* = \frac{u-1}{u-m} > 1, \quad q_3^* = 0 \quad \text{if } m \geq 1, \text{ and}$$

$$q_1^* = 0, \quad q_2^* = \frac{1-d}{m-d} > 1, \quad q_3 = \frac{m-1}{m-d} < 0 \quad \text{if } m \leq 1.$$

This is a signed measure that will be used later, and corresponds to a negative t^* in Equation (13).

Now we consider a European contingent claim $H(S_N)$. We denote by $V_n^*(S_n)$ the superreplication capital defined by Equation (2). One can easily prove (Pliska (1997)) that

$$V_0^*(S_0) = \sup_{t \in (0,1)^k} E^{\mathcal{Q}_t}\{H(S_N)|\mathcal{F}_n\} = \max_{t \in \{0,1\}^k} E^{\mathcal{Q}_t}\{H(S_N)|\mathcal{F}_n\}.$$

The maximiser t^* gives the superreplication capital in this way:

$$V_n^*(S_n) = E^{\mathcal{Q}_{t^*}}\{H(S_N)|\mathcal{F}_n\}. \tag{14}$$

One can easily find t^* by backward recursion. Moreover, if H is convex then $t^* = (1, \dots, 1)$ (see Tessitore and Zabczyk (1996)).

Now we present a theorem in the case when H is a convex function: in fact, this case is much easier than the general case, because under the superreplicating measure $\mathcal{Q}_{t^*}$ the price process S is still Markov, and also $t^* = (1, \dots, 1)$. This allows us to obtain analytic formulae also in this particular case of an incomplete market. In the general case the mathematical tools are slightly different and the notation becomes considerably heavier, so that it is very difficult to write down analytic formulae, which vary from case to case. In the proof of the next theorem we will also use the quantities $V_n^-(S_n)$, $n = N-1, \dots, 0$, defined by

$$V_n^-(S_n) = \inf_{t \in (0,1)^k} E^{\mathcal{Q}_t}\{H(S_N)|\mathcal{F}_n\},$$

which correspond to the lower bound for the arbitrage free prices for H, while the quantities $V_n^*(S_n)$, $n = N-1, \dots, 0$, correspond to the upper bound.

Theorem 2. *Consider a European contingent claim defined by a convex function H. Let $V_n^*(S_n)$, $n = N-1, \dots, 0$, be the superreplication* *capital at time n in Equation (14). If*

$$V_0 > V_0^*(S_0) - \tag{15}$$

$$- \max_{n=0,\dots,N-1} \begin{cases} (q_1^1)^n [V_n^*(S_0 u^n) - V_n^-(S_0 u^n)] \text{ if } \frac{p_1}{q_1^1} \le \frac{p_3}{q_3^1}, \\ \\ (q_3^1)^n [V_n^*(S_0 d^n) - V_n^-(S_0 d^n)] \text{ if } \frac{p_1}{q_1^1} \ge \frac{p_3}{q_3^1}. \end{cases}$$

then, for $n = N-1, \dots, 0$,

$$J(n, S_n, V_n) = \left[\min\left(\frac{p_1}{q_1^1}, \frac{p_3}{q_3^1}\right)\right]^{N-n} \left[V_n^*(S_n) - V_n\right]^+. \tag{16}$$

Moreover, the strategy corresponding to the risk in (16) is given by

$$\hat{\psi}_n = \begin{cases} \psi_n^3 := \dfrac{V_n^*(S_n d) - V_n}{S_n(d-1)} \text{ if } \dfrac{p_1}{q_1^1} \le \dfrac{p_3}{q_3^1}, \\ \\ \psi_n^1 := \dfrac{V_n^*(S_n u) - V_n}{S_n(u-1)} \text{ if } \dfrac{p_1}{q_1^1} \ge \dfrac{p_3}{q_3^1}, \end{cases} \tag{17}$$

which gives a final portfolio value $V_N^{\hat{\psi}}$ such that

$$[H(S_N) - V_N^{\hat{\psi}}]^+ = \tag{18}$$

$$= \begin{cases} \left(\dfrac{1}{q_1^1}\right)^N [V_0^*(S_0) - V_0] \mathbf{1}_{\{\omega_n = u \ \forall n=0,\dots,N-1\}} \text{ if } \dfrac{p_1}{q_1^1} \le \dfrac{p_3}{q_3^1}, \\ \\ \left(\dfrac{1}{q_3^1}\right)^N [V_0^*(S_0) - V_0] \mathbf{1}_{\{\omega_n = d \ \forall n=0,\dots,N-1\}} \text{ if } \dfrac{p_1}{q_1^1} \ge \dfrac{p_3}{q_3^1}. \end{cases}$$

Proof. In order to proceed with the proof, we make the assumption that $m \ge 1$ (for the case $m \le 1$, the proof is similar). We also make the assumption (to be checked later on) that the optimal strategy $\hat{\psi}$ is such that

$$V_n^{\hat{\psi}} \ge V_n^-(S_n) \quad \forall n = 0, \dots, N-1. \tag{19}$$

We proceed by induction starting from $n = N-1$. As in the proof of Theorem 1, the function to be minimized is a linear combination of three affine functions, and the infimum is achieved at one of the three points ψ_{N-1}^1, ψ_{N-1}^3 (defined by Equation (17)) and

$$\psi_{N-1}^2 = \frac{H(S_{N-1} m) - V_{N-1}}{S_{N-1}(m-1)}.$$

It is not difficult to check that the inequality

$$V_{N-1} > E^{Q_0}\{H(S_N)|\mathcal{F}_{N-1}\}, \tag{20}$$

which is true by assumption (19), is equivalent to the condition $\psi^3_{N-1} > \psi^2_{N-1}$ and furthermore, if $V_{N-1} < V^*_{N-1}$ (that is, $J_{N-1}(S_{N-1}, V_{N-1}) > 0$), then one must have $\psi^2_{N-1} < \psi^3_{N-1} < \psi^1_{N-1}$. In fact, if $\psi^3_{N-1} < \psi^1_{N-1}$ did not hold, then we would have that $J(N-1, S_{N-1}, V_{N-1}) = 0$. In order to establish the infimum, we calculate the value of the function j (defined as in Equation (10)) at the three points ψ^i_{N-1}, $i = 1, 2, 3$. In all these calculations, one term in the sum is equal to zero by definition of ψ^i_{N-1}, $i = 1, 2, 3$ and another one is the positive part of a negative quantity (therefore equal to zero). We then easily obtain

$$j(N-1, S_{N-1}, V_{N-1}, \psi^1_{N-1}) = \frac{p_3}{q^3_1}\Big[E^{Q_1}\big\{H(S_N)|\mathcal{F}_{N-1}\big\} - V_{N-1}\Big],$$
$$j(N-1, S_{N-1}, V_{N-1}, \psi^2_{N-1}) = \frac{p_1}{q^*_1}\Big[E^{Q_*}\big\{H(S_N)|\mathcal{F}_{N-1}\big\} - V_{N-1}\Big],$$
$$j(N-1, S_{N-1}, V_{N-1}, \psi^3_{N-1}) = \frac{p_1}{q^1_1}\Big[E^{Q_1}\big\{H(S_N)|\mathcal{F}_{N-1}\big\} - V_{N-1}\Big].$$

Now we prove that $j(N-1, S_{N-1}, V_{N-1}, \psi^2_{N-1}) > j(N-1, S_{N-1}, V_{N-1}, \psi^3_{N-1})$. In fact, using (13), one can easily check that this is equivalent to (20), which is true by assumption. This shows that Equations (16) and (17) are true for $n = N-1$.

We now proceed by induction with respect to n. By arguments similar to those of Theorem 1, we can easily prove Equations (16), (17) and (18). Now we only have to check the initial assumption, that is $V^{\hat\psi}_n \geq V^-_n(S_n)$ for all $n = 0, \dots, N-1$. This is equivalent to saying that $q^n(V^*_n(S_n) - V^{\hat\psi}_n) \leq q^n(V^*_n(S_n) - V^-_n(S_n))$ for all $n = 0, \dots, N-1$. In order to fix the ideas, let us suppose that $p_1/q^1_1 \leq p_3/q^1_3$ (the opposite case follows by a similar argument). Since $\hat\psi$ is the optimal strategy, we have

$$J(0, S_0, V_0) = \left(\frac{p_1}{q^1_1}\right)^N (V^*_0(S_0) - V_0)^+ = E[J(n, S_n, V^{\hat\psi}_n)]. \tag{21}$$

Since by Equation (18) $V_n < V^*_n(S_n)$ only in the event $\{\omega_i = u \quad \forall i = 0, \dots, n-1\}$, we have

$$\begin{aligned} J(0, S_0, V_0) &= 0 + p^n_1 J(n, S_n, V^{\hat\psi}_n)|_{\omega_i = u\ \forall i = 0,\dots,n-1} = \\ &= p^n_1 \left(\frac{p_1}{q^1_1}\right)^{N-n} (V^*_n(S_n) - V^{\hat\psi}_n)|_{\omega_i = u\ \forall i = 0,\dots,n-1}. \end{aligned} \tag{22}$$

By comparing Equations (21) and (22), it follows that $(q_1^1)^n(V_n^*(S_n) - V_n^{\hat{\psi}}) = V_0^*(S_0) - V_0$. So we have to check that $V_0^*(S_0) - V_0 \leq (q_1^1)^n(V_n^*(\hat{S}_n) - V_n^-(\hat{S}_n))$ for all $n = 0, \ldots, N-1$, which is equivalent to Equation (15). $\square$

Remark 2. We notice that in order to have an explicit formula one of our assumptions is that the initial capital V_0 is not too small. In particular, we saw that this assumption corresponds to the fact that, at each time step $n = 0, \ldots, N$, the capital $V_n^{\hat{\psi}}$ along the optimal strategy $\hat{\psi}$ is always greater than $V_n^-(S_n)$. In order for this to be verified, it is sufficient to check it only on the worst path of the risky asset, namely the only one in which the shortfall is positive.

Remark 3. Like Theorem 1, Theorem 2 also leads to a generalisation of the results in Runggaldier *et al.* (2001): in fact, the solution (both in the minimal shortfall as in the optimal strategy) turns out to be equal to the solution in the case of a binomial model which is obtained by setting equal to zero the probability of one of the three possible states of nature at each step. This state of nature is such that the expectation of the contingent claim under the resulting probability is equal to the superreplication capital needed at each step.

Remark 4. When H is convex we find a nice parallel between this case and superreplication in stochastic volatility models in continuous time (see Avellaneda *et al.* (1995), El Karoui *et al.* (1998)). In fact, in both cases the superreplication capital is equal to the expectation of the claim under a measure which is not equivalent to the real world probability, and this measure is extremal in different senses: in the stochastic volatility case, it is obtained by taking the supremum of the admissible volatilities in an interval $[\sigma_{\min}, \sigma_{\max}]$ (see El Karoui *et al.* (1998)), while here it is obtained by taking the supremum of the weights at the extremal points of the support $\{d, m, u\}$. This comes as no surprise, because a trinomial tree can be used to approximate a stochastic volatility model (see Avellaneda *et al.* (1995)).

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