Invariant measures for a Langevin equation describing forward rates in an arbitrage free market.

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Abstract

In this article the forward rate equation proposed in [6] is analysed; the equation is studied in different Hilbert spaces, namely $H^1_\gamma([0,+\infty))$ and $H^1([0,+\infty))$, and in the locally convex space $W^{1,1}_{loc}([0,+\infty))$, which is his natural space; for each of these spaces, explicit mild solutions and sufficient conditions for the existence of invariant measures are presented. In the case of the Hilbert spaces, all the possible invariant measures are characterized.

1 Introduction

This work starts from the interest rates term structure model proposed by Musiela in 1993 [6]. In that work, Musiela proposed a forward rate model in which $r(t,x)$ represented the rate at which one can enter a contract (to borrow or to lend) at time t for a short period of time at $t+x$, with $x \geq 0$. This approach is similar to the Heath - Jarrow - Morton model treated in [4], with the difference that the last one uses an f such that $f(t,t+x) = r(t,x)$. As in [4], the work furnished an equation for the forward rate, provided we are under the measure for which the actualized bond price is a martingale. Making right hypotheses, the problem is taken back to a Langevin equation taking values in a suitable function space ($W^{1,1}_{loc}$ if no other hypotheses are given). The explicit solution of the equation was then given and conditions under which an invariant measure exist were presented. But these results were given without proof and the function spaces where the solutions and the invariant measures lie were not explicitly indicated. Moreover, whether such invariant measure is unique, and, in the opposite case, a characterization of all invariant measures, are questions not discussed in [6].

This article deals with the problem treated by Musiela; in particular, it studies the Langevin equation in different spaces contained in $W^{1,1}_{loc}([0,+\infty))$. For each space the problems of the existence of a mild solution, of finding its law and of the existence (and possibly of the uniqueness) of an invariant measure are studied. We obtain, in particular, the results already given in [6]; besides, in some of our spaces, we find other invariant measures than the one found in [6] and we obtain a complete characterization of all the invariant measures. Paragraph 2 is a presentation of the Musiela model. Paragraph 3 presents the hypotheses under which the equation becomes a Langevin equation and contains the formulation of the problem solved in this article. Paragraph 4 gives some general theory about the Langevin equation in separable Hilbert spaces and an abstract theorem about existence of invariant measures for a Langevin equation in separable Hilbert spaces. For this paragraph and for the three following, we deal with the theory of stochastic integration as presented in [2]. In paragraph 5, the Musiela equation is studied in the Hilbert space $H^1_\gamma([0,+\infty))$, which is a linear subspace of $W^{1,1}_{loc}([0,+\infty))$; we find a wide range of invariant measures, showing also that all of them have equal plausibility, that is there is not a privileged measure. In paragraph 6 the same equation is studied in $H^1([0,+\infty))$, and we find out that there exist only one invariant measure in this space. In paragraph 7 similar results in two other Hilbert spaces, namely $L^2_\gamma([0,+\infty))$ and $L^2([0,+\infty))$, are given for a mathematical curiosity. In paragraph 8 the equation is studied in the general case, that is, lacking other hypotheses, the equation has to be solved in $W^{1,1}_{loc}([0,+\infty))$. For this, we embed $W^{1,1}_{loc}([0,+\infty))$ in the Schwartz distributions space and we use the stochastic calculus theory presented in [5] for this space. Also, here we find out that there are infinitely many invariant gaussian measures, Finally, there is an appendix, in which the semigroup results that were useful during the article are stated and proved.

Now a word about the mathematical modelling. Which space is the right one in the applications is not yet clear. In some cases, it seems natural to deal with diffusion terms $\tau(x)$ which decay to zero as x tends to infinity. In such a case, the space H^1 may be appropriate and we find that the system displays a unique asymptotic regime (governed by the unique invariant measure we find in that space). In other situations, diffusion terms having finite limit different from zero are natural, so that we use the spaces H^1_γ and $W^{1,1}_{loc}$. In such case the asymptotic regime may depend upon the initial data $r(0,x)$ and a large variety of asymptotic evolutions are possible.

Just a word about the mathematical techniques. It is frequent, in works dealing with Ornstein - Uhlenbeck processes and Langevin equations in spaces like $\mathcal{D}^*$, to start with the Ornstein - Uhlenbeck process, and then to prove that it satisfies a Langevin equation, where the corresponding semigroup satisfies some properties (namely, it is an equicontinuous C^0 semigroup) (see for example [1]). Here, we started from a concrete problem and we obtained an equation to be satisfied by a process, so we built the process given the equation following the most logical way to arrive to the solution. The problem here was that our semigroup was not an equicontinuous C^0 semigroup, so we had to introduce a new definition, namely that of a locally equicontinuous C^0 semigroup. This is a property which our semigroup satisfies, and so all the machinery works.

A final note: the original model of Musiela uses a 1-dimensional Brownian motion, but the results found in this case are trivially extendible to the case of a k-dimensional Brownian motion. An extension to the case of a generic infinite dimensional Brownian motion can be difficult, expecially in the general case $r_t \in W^{1,1}_{loc}$. For this reason, in this work all the results are stated using a k-dimensional Brownian motion, but the proofs and the calculations in general are carried out using a 1-dimensional Brownian motion in order not to lose the main ideas behind too many calculations.

2 Presentation of the Musiela model

Let's introduce now the Musiela interest rate term structure model. We have a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with the filtration $(\mathcal{F}_t)_{t\geq 0}$. We suppose to have only a single bond in the market, and we suppose that the price at time t of a bond expiring at time T is given by the process $(P(t,T))_{t\geq 0}$. We suppose to have a two parameters process $(r(t,x))_{t,x\geq 0}$, called **forward rate** process, in which $r(t,x)$ represents the rate at which at time t one can enter a contract (to borrow or lend money) at time $T = t + x$, for a short period of time. Then the price at time t of a bond expiring at time $T = t + x$ is given by:

$$P(t,T) = P(t,t+x) = \exp\left(-\int_0^x r(t,u)\ du\right)$$

We call **spot rate** process the process $(r(t,0))_t$; it represents the rate at which at time t one can enter a contract expiring immediately after. We also call **price progress of the savings account** the process $(\beta(t))_t$, given by:

$$\beta(t) = \exp\left(\int_0^t r(u,0)\ du\right)$$

The **actualized price** at time t of a bond expiring at time $t + x$ is given by:

$$\begin{aligned} \tilde{P}(t,T) &= \frac{P(t,T)}{\beta(t)} = \exp\left(\int_0^t r(u,0)\ du - \int_0^{T-t} r(t,u)\ du\right) = \\ &= \exp\left(\int_0^t r(u,0)\ du - \int_0^x r(t,u)\ du\right) \end{aligned}$$

Now we add the hypotheses that there exists a k-dimensional standard brownian motion $(W_t)_t$ adapted to $(\mathcal{F}_t)_t$, and there exist two two-parameter progressively measurable processes $(\alpha(t,x))_{t,x\geq 0}$, $(\tau(t,x))_{t,x\geq 0}$ such that $\forall x \geq 0, (\alpha(t,x))_t$ has trajectories in $L^1([0,+\infty);\mathbf{R})$, $(\tau^*(t,x))_t$ has trajectories in $L^2([0,+\infty);\mathbf{R}^k)$ and such that the forward rate satisfies the following stochastic differential equation:

$$\begin{cases} dr(t,x) = \alpha(t,x)\ dt + \tau^*(t,x)\ dW_t \\ r(0,x) \text{ given} \end{cases} \tag{1}$$

A natural thing to ask is that the actualized bond price process $(\tilde{P}(t,T))_t$ is a martingale under a suitable measure $\mathbf{P}$. To this purpose, we cite the:

Theorem 1 *: if we model the forward rate with an equation of the form (1), then the following facts are equivalent:*

1) $\forall T > 0$ *the process* $(\tilde{P}(t,T))_t$ *is a local martingale with rispect to* $\mathbf{P}$.

2) the process $(r_t(x))_t$ *is such that* $\forall t \in [0, +\infty))$ *the application* $x \to r(t,x)$ $\mathbf{P}$*-a.s. belongs to* $AC([0,+\infty))$ *and:*

$$\alpha(t,x) = \frac{\partial}{\partial x}\left(r(t,x) + \frac{1}{2}\left| \int_0^x \tau(t,u)\ du \right|^2 \right)$$

For a proof see [6].

□

The forward rate model obtained in this way is then the following:

$$\begin{cases} dr(t,x) = \left(\frac{\partial}{\partial x} r(t,x) + \tau^*(t,x) \int_0^x \tau(t,u)\ du\right)\ dt + \tau^*(t,x)\ dW_t \\ r(0,x) \text{ given} \end{cases} \tag{2}$$

Now we will study this model using more powerful instruments. We now view the process $(r(t,x))_{t,x}$ no more as a real two parameter process, but as a one parameter process taking values in the absolutely continuous functions space on $([0,+\infty))$, which will be identified with the Sobolev space $W^{1,1}_{loc}([0,+\infty))$, and we write the equation (2) as a stochastic differential equation taking values in $W^{1,1}_{loc}$.

Taking:

$$A = \frac{\partial}{\partial x}$$

$$\tau_t(x) = \tau(t,x)$$

$$c_t(x) = \tau^*_t(x) \int_0^x \tau_t(u)\ du$$

then equation (2) becomes the following equation taking values in $W^{1,1}_{loc}$:

$$\begin{cases} dr_t = (Ar_t + c_t)\ dt + \tau^*_t\ dW_t \\ r_0 \in W^{1,1}_{loc} \end{cases} \tag{3}$$

If we make the further hypotheses that $(\mathcal{F}_t)_t$ is the completion of the natural filtration of the brownian motion $(W_t)_t$, and that $|\tau_t(x)| \leq M(T)$ $\forall t, x$ such that $t + x \leq T$ $\mathbf{P}$-a.s., then the process $(\tilde{P}(t,T))_t$ is a martingale; then the hypotheses of the martingale representation theorem hold, and so every integrable random variable (that is every contingent claim) can be represented as the sum of his expectation, which is the arbitrage free price of the claim, and of a stochastic integral with respect to $(W_t)_t$. Besides, we can construct a self-financing portfolio strategy based on $P(.,T)$ and on $\beta(.)$, which simulates the claim (see [6]).

3 Formulation of the problem

Now we make some hypotheses under which this equation will have explicit solutions. We suppose that the process $(\tau_t)_t$ is identically equal to a real k-valued deterministic function $\tau(x)$ belonging to $W^{1,1}_{loc}$. Then the equation becomes:

$$\begin{cases} dr_t = (Ar_t + c)\ dt + \tau^*\ dW_t \\ r_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H) \end{cases} \tag{4}$$

This is a Langevin equation, which, under right hypotheses, has an explicit solution.

We now want to study the equation. We start by considering two particular separable Hilbert spaces contained in $W^{1,1}_{loc}$. We use the theory of the stochastic integration in separable Hilbert spaces contained in [2] which is somewhat complete. In particular, we will study the equation in the spaces:

$$H^1_\gamma([0,+\infty)) = W^{1,2}_\gamma([0,+\infty)) = \Big\{u : [0,+\infty) \to \mathbf{R} \mid u \in L^2_\gamma([0,+\infty)), u' \in L^2_\gamma([0,+\infty))\Big\}$$

$$H^1([0,+\infty)) = W^{1,2}([0,+\infty)) = \Big\{u : [0,+\infty) \to \mathbf{R} \mid u \in L^2([0,+\infty)), u' \in L^2([0,+\infty))\Big\}$$

where u' is the weak derivative of u with respect to x, and:

$$L^2_\gamma([0,+\infty)) = \left\{u : [0,+\infty) \to \mathbf{R} \;\middle|\; u \text{ meas. and s.t. } \int_0^{+\infty} u^2(x)e^{-\gamma x}\ dx < +\infty\right\}$$

$$L^2([0,+\infty)) = \left\{u : [0,+\infty) \to \mathbf{R} \;\middle|\; u \text{ meas. and s.t. } \int_0^{+\infty} u^2(x)\ dx < +\infty\right\}$$

H^1_γ is a Hilbert space with the scalar product:

$$< f, g >_{H^1_\gamma} = \gamma \int_0^{+\infty} f(x)g(x)e^{-\gamma x}\ dx + \gamma \int_0^{+\infty} f'(x)g'(x)e^{-\gamma x}\ dx \tag{5}$$

and H^1 is a Hilbert space with the scalar product:

$$< f, g >_{H^1} = \int_0^{+\infty} f(x)g(x)\ dx + \int_0^{+\infty} f'(x)g'(x)\ dx \tag{6}$$

Then, after having cited, only for a mathematical curiosity, similar results in L^2_γ and in L^2, we will study the equation in his natural environment, that is in the space:

$$W^{1,1}_{loc}([0,+\infty)) = \Big\{u : [0,+\infty) \to \mathbf{R} \mid u_{[0,T]} \in L^1([0,T)), u'_{[0,T]} \in L^1([0,T))\ \forall T > 0\Big\}$$

In order to study the equation, we will embed $W^{1,1}_{loc}$ in the Schwarz distributions space $\mathcal{D}^*$, dual of the space $C^\infty_0((-\infty,+\infty))$, which is a locally convex space with respect to some Hilbertian norms (for a detailed discussion, see the appendix); in doing this, we will follow the theory presented in [5].

In all the spaces cited above we will examine the following problems:

- to find an explicit mild solution

- to find the marginal law of the solutions r_t $\forall t > 0$ and the joint law of the process $(r_t)_{t>0}$

- existence, uniqueness (when it holds) and characterization of the invariant measures for the solution process

4 Preliminaries about Langevin equations in Hilbert spaces

Now we give a bit of general theory about the Langevin equation on separable Hilbert spaces.

We suppose we have a separable Hilbert space H, such that $r_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H)$, $c \in H$, that $A : D(A) \to H$ is the infinitesimal generator of a C^0 semigroup of linear continuous operators (see appendix), that $(W_t)_t$ is a Q-Wiener process in H, that is a $(\mathcal{F}_t)_t$-stochastic process such that:

1) $W_0 = 0$

2) $(W_t)_t$ has continuous trajectories

3) $\forall s < t$, $W_t - W_s$ is independent from $\mathcal{F}_s$

4) $\forall s < t$, $W_t - W_s$ has gaussian law equal to $N(0, (t-s)Q)$, where $Q : H \to H$ is linear, bounded, symmetric, positive and such that $\mathrm{Tr}\ Q < +\infty$

and that, if we call $Q_t = S_t Q S_t^*$, then $\int_0^t \mathrm{Tr}\ Q_u \ du < +\infty$, then the equation

$$\begin{cases} dr_t = (Ar_t + c)\ dt + dW_t \\ r_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H) \end{cases} \tag{7}$$

has exactly one mild solution, which is the Ornstein-Uhlenbeck process in the Hilbert space H:

$$r_t = S_t r_0 + \int_0^t S_{t-u} c\ du + \int_0^t S_{t-u}\ dW_u$$

If r_0 is a gaussian random variable, then the solution is a gaussian process with functional mean:

$$E[r_t] = S_t r_0 + \int_0^t S_{t-u} c\ du \tag{8}$$

and functional covariance (if $t \le v$):

$$Cov(r_t, r_v) = S_v Cov(r_0, r_0) S_t^* + \int_0^t S_{v-u} Q S_{t-u}^*\ du \tag{9}$$

This means that $\forall f, g \in H$:

$$E[< f, r_t >_H] = < f, S_t r_0 >_H + < f, \int_0^t S_{t-u} c\ du >_H$$

$$
\begin{aligned}
Cov(<f,r_t>_H,<g,r_v>_H) &= <(S_v Cov(r_0,r_0)S_t^*)f,g>_H + \int_0^t <S_{v-u}QS_{t-u}^* f,g>_H= \\
&= <Cov(r_0,r_0)S_t^* f, S_v^* g>_H + \int_0^t <QS_{t-u}^* f, S_{v-u}^* g>_H \; du
\end{aligned}
$$

For a proof of these facts, see [2].

Applying these results to our equation, since $A = \frac{\partial}{\partial x}$, the natural semigroup is the translation semigroup $(S_t)_t$ such that:

$$
S_t f(x) = f(t+x), \qquad t,x \geq 0
$$

Moreover, the Wiener process is of the kind $(\sum_{i=1}^k \tau_i W_t^i)_t$, where $\tau \in H^k$ and $(W_t)_t$ is a real k-dimenional Wiener process; then $Q = \sum_{i=1}^k \tau_i \otimes \tau_i$, in the sense that:

$$
Qf = \sum_{i=1}^k <\tau_i, f>_H \tau_i \qquad \forall f \in H
$$

Then Tr $Q = \sum_{i=1}^k \|\tau_i\|_H^2 < +\infty$. The solution becomes:

$$
\begin{aligned}
r_t(x) &= S_t r_0(x) + \int_0^t S_{t-u} c(x) \; du + \sum_{i=1}^k \int_0^t S_{t-u} \tau_i(x) \; dW_u^i = \\
&= r_0(x+t) + \sum_{i=1}^k \int_0^t \tau_i(x+t-u) \left(\int_0^{x+t-u} \tau_i(v) \; dv \right) \; du + \\
&\quad + \int_0^t \tau_i(x+t-u) \; dW_u^i = \\
&= r_0(x+t) + \sum_{i=1}^k \int_x^{x+t} \tau_i(u) \int_0^u \tau_i(v) \; dv \; du + \sum_{i=1}^k \int_0^t \tau_i(x+t-u) \; dW_u^i = \\
&= r_0(x+t) + \sum_{i=1}^k \int_0^{x+t} \tau_i(u) \int_0^u \tau_i(v) \; dv \; du + \\
&\quad - \sum_{i=1}^k \int_0^x \tau_i(u) \int_0^u \tau_i(v) \; dv \; du + \sum_{i=1}^k \int_0^t \tau_i(x+t-u) \; dW_u^i = \\
&= r_0(x+t) + \frac{1}{2} \sum_{i=1}^k \left(\left(\int_0^{x+t} \tau_i(u) \; du \right)^2 - \left(\int_0^x \tau_i(u) \; du \right)^2 \right) + \\
&\quad + \sum_{i=1}^k \int_0^t \tau_i(x+t-u) \; dW_u^i
\end{aligned}
$$

The solution is a gaussian process with functional mean (8) and functional covariance (9); this means that $\forall f,g \in H$:

$$
E[<r_t,f>_H] = <r_0(.+t),f>_H + \frac{1}{2} \sum_{i=1}^k < \left(\left(\int_0^{.+t} \tau_i(u) \; du \right)^2 - \left(\int_0^{.} \tau_i(u) \; du \right)^2 \right), f>_H
$$

$$
\begin{aligned}
Cov(<r_t, f>_H, <r_v, g>_H) &= <S_v Cov(r_0, r_0) S_t^* f, g>_H + \\
&+ \sum_{i=1}^{k} < \left(\int_0^s \tau_i(.+t-u) \otimes \tau_i(.+v-u)\ du \right) f, g>_H = \\
&= <Cov(r_0, r_0) S_t^* f, S_v^* g>_H + \\
&+ \sum_{i=1}^{k} \int_0^t << \tau_i(.+t-u), f>_H \tau_i(.+v-u), g>_H\ du = \\
&= <Cov(r_0, r_0) S_t^* f, S_v^* g>_H + \\
&+ \sum_{i=1}^{k} \int_0^t < \tau_i(.+t-u), f>_H < \tau_i(.+v-u), g>_H\ du
\end{aligned}
$$

In some applications it can be useful to know only mean and covariance of particular rates and not necessarily all the forward rates curve (for example, to know the spot rate at time t, it is sufficient to know only $r_t(0)$). This can be done with the theory given before: if the Hilbert space H is a subset of the space of the continuous functions $C^0([0, +\infty))$, then $\forall t, v, x, y \geq 0$:

$$
E[r_t(x)] = E[r(0, x+t)] + \frac{1}{2} \sum_{i=1}^{k} \left(\left(\int_0^{x+t} \tau_i(u)\ du \right)^2 - \left(\int_0^{x} \tau_i(u)\ du \right)^2 \right) \tag{10}
$$

$$
Cov(r_t(x), r_v(y)) = Cov(r_0(t+x), r_0(v+y)) + \sum_{i=1}^{k} \int_0^t \tau_i(x+t-u) \tau_i(y+v-u)\ du \tag{11}
$$

To deal with the invariant measures problem, we will need the following theorem:

Theorem 2 *: if we call $Q_t = \int_0^t S_u Q S_u^*\ du$, and the following condition is satisfied:*

1) $\sup_{t \geq 0} \mathrm{Tr}\ (Q_t) < +\infty$

together with one of the following:

2) it exists an invariant measures ν for the equation:

$$
dZ_t = (AZ + c)\ dt
$$

3) if $c \in Im(A)$, and it exists an invariant measure ν for the equation:

$$
dZ_t = AZ\ dt
$$

then it exists an invariant measure for the equation (7), and in particular:

- if (2) is satisfied, then every invariant measure is of the form:

$$
\mu = \nu * N(0, Q_\infty)
$$

- if (3) is satisfied, then every invariant measure is of the form:

$$\mu = \nu * \delta_b * N(0, Q_\infty)$$

where $b \in D(A)$ is a vector in H such that $Ab = -c$.

For a proof see [3].

□

5 The forward rate equation in the space H^1_γ

Now let's study the case in which the Hilbert space is $H^1_\gamma([0, +\infty))$, endowed with the scalar product (5).

Theorem 3 *: if $\tau_i \in H^1_\gamma \cap H^1 \cap L^4_\gamma$ $\forall i = 1, ..., k$ and $r_0 \in L^2_\gamma$, then it exist a unique mild solution of the equation (4), given by:*

$$r_t(x) = r_0(x+t) + \frac{1}{2}\sum_{i=1}^{k}\left(\left(\int_0^{x+t} \tau_i(u)\ du\right)^2 + \left(\int_0^{x} \tau_i(u)\ du\right)^2\right) + \sum_{i=1}^{k}\int_0^t \tau_i(x+t-u)\ dW^i_u \tag{12}$$

The solution is a gaussian process with functional mean (8) and functional covariance (9).

Proof: first we must check that the equation is well defined in this space. We then impose the conditions $r_0 \in H^1_\gamma$, $\tau \in H^1_\gamma$, $c \in H^1_\gamma$ and $\int_0^t \mathrm{Tr}\ (S_u Q S^*_u)\ du < +\infty$ $\forall t$. The calculations for these two last conditions are somewhat hard, but we can find sufficient conditions for this to happen. Later we will find out that these conditions are necessary for the existence of an invariant measure.

First of all we must check that the weak derivative of c is well defined:

$$\frac{\partial}{\partial x}c(x) = \frac{\partial \tau}{\partial x}(x)\int_0^x \tau(u)\ du + \tau^2(x)$$

is well defined, because $\tau \in H^1_\gamma$; let's check now that both c and c' are in L^2_γ: if we impose that $\tau \in L^2([0, +\infty))$, then we have that:

$$\begin{aligned}
\int_0^{+\infty} c^2(x)\gamma e^{-\gamma x}\ dx &= \int_0^{+\infty} \tau^2(x)\gamma e^{-\gamma x}\left(\int_0^x \tau(u)\ du\right)^2\ dx \leq \\
&\leq \int_0^{+\infty} \tau^2(x)\gamma e^{-\gamma x} x \int_0^x \tau^2(u)\ du\ dx \leq \\
&\leq \int_0^{+\infty} \tau^2(x)\gamma e^{-\gamma x} x \int_0^{+\infty} \tau^2(u)\ du\ dx \leq \\
&\leq \|\tau\|^2_{L^2}\int_0^{+\infty} \tau^2(x)\|\gamma x e^{-\gamma x}\|_{C^0}\ dx \leq \|\gamma x e^{-\gamma x}\|_{C^0}\|\tau\|^4_{L^2}
\end{aligned}$$

and then $c \in L^2_\gamma$. Now let's check the weak derivative:

$$\begin{aligned}
\|c'\|_{L^2_\gamma} &\leq 2\left\|\frac{\partial \tau}{\partial x}\int_0^{\cdot} \tau(u)\ du\right\|_{L^2_\gamma} + 2\|\tau^2\|_{L^2_\gamma} = \\
&= 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\int_0^x \tau(u)\ du\right)^2 \gamma e^{-\gamma x}\ dx + 2\|\tau\|^2_{L^4_\gamma} = \\
&= 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2\left(\int_0^x \tau(u)\ du\right)^2 \gamma e^{-\gamma x}\ dx + 2\|\tau\|^2_{L^4_\gamma} \leq \\
&\leq 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 x\int_0^x \tau^2(u)\ du\ \gamma e^{-\gamma x}\ dx + 2\|\tau\|^2_{L^4_\gamma} \leq \\
&\leq 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 \int_0^{+\infty} \tau^2(u)\ du\ x\gamma e^{-\gamma x}\ dx + 2\|\tau\|^2_{L^4_\gamma} \leq \\
&\leq 2\|\tau\|^2_{L^2}\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 x\gamma e^{-\gamma x}\ dx + 2\|\tau\|^2_{L^4_\gamma} \leq \\
&\leq 2\|\tau\|^2_{L^2}\left\|\frac{\partial \tau}{\partial x}\right\|^2_{L^2}\|x\gamma e^{-\gamma x}\|_{C^0} + 2\|\tau\|^2_{L^4_\gamma}
\end{aligned}$$

Then $\tau \in L^4_\gamma, \tau' \in L^2$ and $\tau \in L^2$ are sufficient conditions for $c \in H^1_\gamma$ Then one can verify that $A : D(A) \to H$ is the infinitesimal generator of the C^0 semigroup $(S_t)_t$, and has the domain:

$$D(A) = H^2_\gamma([0,+\infty)) = W^{2,2}_\gamma([0,+\infty)) = \{u \in L^2_\gamma \mid u' \in L^2_\gamma, u'' \in L^2_\gamma\}$$

where u' indicates the first weak derivative, and u'' the second weak derivative. For a proof, see the appendix.

Finally, we have to check that:

$$\begin{aligned}
\int_0^t \mathrm{Tr}\ (S_u Q S_u^*)\ du &= \int_0^t \|\tau(.+u)\|^2_{H^1_\gamma}\ du \leq \int_0^{+\infty}\|\tau(.+u)\|^2_{H^1_\gamma}\ du = \\
&= \int_0^{+\infty}\int_0^{+\infty}\gamma(\tau^2(x+u) + \tau'^2(x+u))e^{-\gamma x}\ dx\ du = \\
&= \int_0^{+\infty}\gamma e^{-\gamma x}\int_0^{+\infty}(\tau^2(x+u) + \tau'^2(x+u))\ du\ dx = \\
&= \int_0^{+\infty}\gamma e^{-\gamma x}\int_x^{+\infty}(\tau^2(u) + \tau'^2(u))\ du\ dx \leq \int_0^{+\infty}\gamma e^{-\gamma x}\|\tau\|^2_{H^1}\ dx
\end{aligned}$$

We notice that the condition $\tau \in H^1$ is sufficient to have $\int_0^t \mathrm{Tr}\ (S_u Q S_u^*)\ du < +\infty$.

□

Now we analize the problem of finding an invariant measure. First of all we estimate $\sup \mathrm{Tr}\ Q_t$:

$$\sup_{t\geq 0} \mathrm{Tr}\ Q_t = \int_0^{+\infty}\|\tau(.+u)\|^2_{H^1_\gamma}\ du \leq \sum_{i=1}^k \int_0^{+\infty}\gamma e^{-\gamma x}\|\tau\|^2_{H^1}\ dx$$

We notice that the condition $\tau \in H^1$ is sufficient to guarantee that $\sup \mathrm{Tr}\, Q_t < +\infty$. Conversely, it is also necessary, since the integral to the last member but one converges only if the integrand function is almost surely finite, that is whether $\int_x^{+\infty} (\tau^2(u) + \tau'^2(u))\ du$ exists for almost every x.

Let's check now the condition (2) of the proposition (2). To find an invariant measure for the equation

$$dZ_t = (AZ_t + c)\ dt \tag{13}$$

we search for a solution of the partial differential equation:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} + c$$

independent from time. We try to solve the differential equation:

$$\frac{\partial z}{\partial x} + c = 0$$

in the space H^1_γ; a solution of this equation is:

$$z(x,t) = z^*(x) = -\int_0^x c(x)\ dx = -\int_0^x \tau(u) \int_0^u \tau(v)\ dv\ du$$

Let's check if $z^*(x) \in H^1_\gamma$. First of all let's verify that $z^*(x) \in L^2_\gamma$:

$$\begin{aligned} |z^*(x)| &= \left| \int_0^x \tau(u) \int_0^u \tau(v)\ dv\ du \right| \leq \int_0^x |\tau(u)| \int_0^u |\tau(v)|\ dv\ du \leq \\ &= \frac{1}{2} \left(\int_0^x |\tau(u)|\ du \right)^2 \leq \frac{1}{2} x \int_0^x |\tau(u)|^2\ du \leq x \|\tau\|^2_{L^2_\gamma} \end{aligned}$$

so $\|z^*\|_{L^2_\gamma} \leq \|\tau\|^2_{L^2_\gamma} \|x\|_{L^2_\gamma}$, and $z^* \in L^2_\gamma$.

Besides, z^* has the weak derivative equal to $-c$, which is in H^1, and so in L^2; then $z^* \in H^1$, so, if the initial data has law δ_{z^*}, then the soluzion has law δ_{z^*} $\forall t \geq 0$, and then there exists an invariant measure for the equation (13).

We also notice that also the constant functions belong to H^1_γ, and so every initial data of the kind $\delta_{z^*+d}, d \in \mathbf{R}$ generates an invariant measure, and the corresponding process has a marginal law δ_{z^*+d} $\forall t \geq 0$.

We notice that we have infinitely many invariant measures. A possible characterization theorem of all would be very intricate. In fact, if we take for (13) an initial data $f \in H^1_\gamma$ periodical with period T, then the solution for (Z) would be $z(x,t) = f(t+x)$. Then, if we define the random variable $\pi : ([0,T], \mathcal{B}([0,T])) \in H^1_\gamma$ such that $\pi(t) = f(.+t)$ and we pose the uniform density on $[0,T]$, then π induces a measure μ_f on H^1_γ such that the initial data has law μ_f, then we obtain an invariant measure with marginal law μ_f $\forall t \geq 0$. We stop here and we collect the results we obtain in the following:

Theorem 4 : *given the equation (4) in the Hilbert space H^1_γ, a necessary and sufficient condition to have an invariant measure is:*

$$\tau_i \in H^1_\gamma([0,+\infty)) \cap H^1([0,+\infty)) \ \forall i = 1, ..., k$$

and there exist infinitely many invariant measures. In particular, the measures of the kind:

$$\delta_{z^*+c} * N(0, Q_\infty), c \in \mathbf{R}$$

$$\mu_f * N(0, Q_\infty), f \in H^1_\gamma \text{ periodical}$$

are invariant measures, where:

$$Q_\infty = \sum_{i=1}^{k} \int_0^{+\infty} \tau_i(.+u) \otimes \tau_i(.+u)\ du$$

$$z^*(x) = -\sum_{i=1}^{k} \int_0^x \tau_i(u) \int_0^u \tau_i(v)\ dv\ du$$

$f \in H^1_\gamma$ is periodical with period T and μ_f is the measure induced by the uniform density on $[0, T]$ via the application $t \to f(t+.)$.

□

Now a problem arises: from where do we get all these invariant measures? More precisely, if we take a general initial data, will there be let's say a privileged invariant measure to which this initial data would converge in some way? Or do all these measures have equal dignity?

Let's make a try (with $k = 1$): let's take an initial data with a degenerate law concentrated on r_0, and let's see what happens to the marginal law of the solution $r_t(\mathbf{P})$. The solution r_t has functional mean:

$$\begin{aligned} E[r_t](.) &= E[S_t r_0](.) + E\left[\int_0^t S_{t-u} c\ du\right](.) = \\ &= r_0(.+t) + \int_0^t \tau(.+t-u) \int_0^{.+t-u} \tau(v)\ dv\ du = \\ &= r_0(.+t) + \int_.^{.+t} \tau(u) \int_0^u \tau(v)\ dv\ du \end{aligned}$$

and functional variance:

$$Var[r_t] = \int_0^t \tau(.+t-u) \otimes \tau(.+t-u)\ du = \int_0^t \tau(.+u) \otimes \tau(.+u)\ du$$

If r_0 has a finite limit for $x \to +\infty$, (and we call this limit $r_0(\infty)$), r_0' has limit equal to 0 for $x \to +\infty$, and $\tau \in H^1_\gamma \cap H^1$, then $\forall f, g \in H^1_\gamma$ we have:

$$\begin{aligned}
\lim_{t\to+\infty} \quad < E[r_t], f >_{H^1_\gamma} = & \lim_{t\to+\infty} \left(\int_0^{+\infty} f(x) r_0(x+t) e^{-\gamma x} \, dx + \right.\\
& + \int_0^{+\infty} f'(x) r_0'(x+t) \gamma e^{-\gamma x} \, dx + \\
& + \int_0^{+\infty} f(x) \left(\int_x^{x+t} \tau(u) \int_0^u \tau(v) \, dv \, du \right) e^{-\gamma x} \, dx + \\
& \left. + \int_0^{+\infty} f'(x) \left(\tau(x+t) \int_0^{x+t} \tau(v) \, dv - \tau(x) \int_0^x \tau(v) \, dv \right) e^{-\gamma x} \, dx \right) = \\
& =< r_0(\infty), f >_{H^1_\gamma} + \gamma \left(\int_0^{+\infty} f(x) \left(\int_x^{+\infty} \tau(u) \int_0^u \tau(v) \, dv \, du \right) e^{-\gamma x} \, dx + \right.\\
& \left. + \int_0^{+\infty} f'(x) \left(-\tau(x) \int_0^x \tau(v) \, dv \right) e^{-\gamma x} \, dx \right) = \\
& =< r_0(\infty), f >_{H^1_\gamma} + < \int_.^{+\infty} \tau(u) \int_0^u \tau(v) \, dv \, du, f >_{H^1_\gamma}
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to+\infty} < Var[r_t] f, g >_{H^1_\gamma} &= \lim_{t\to+\infty} \int_0^t < \tau(.+u), f >_{H^1_\gamma} < \tau(.+u), g >_{H^1_\gamma} \, du = \\
&= \int_0^{+\infty} < \tau(.+u), f >_{H^1_\gamma} < \tau(.+u), g >_{H^1_\gamma} \, du
\end{aligned}$$

and so we have:

$$\begin{aligned}
\lim_{t\to+\infty} E[r_t](.) &= r_0(\infty) + \int_.^{+\infty} \tau(u) \int_0^u \tau(v) \, dv \, du = \\
&= r_0(\infty) + \frac{1}{2} \left(\left(\int_0^{+\infty} \tau(u) \, du \right)^2 - \left(\int_0^{.} \tau(u) \, du \right)^2 \right)
\end{aligned}$$

$$\lim_{t\to+\infty} Var[r_t] = \int_0^{+\infty} \tau(.+u) \otimes \tau(.+u) \, du$$

so the functional mean and covariance of the marginal law of the solution process converge to the functional mean and covariance of an invariant law, and so, since we are dealing with normal laws, this means that the marginal law of the solution converges weakly to an invariant measure. The particular invariant measure to which the solution law converge is determined by $r_0(\infty)$. In fact, if we build a suitable initial function r_0, it is possible to reach any invariant gaussian law we found in the last theorem; then it seems that there are no privileged invariant measures.

6 The forward rate equation in the space H^1

Now, let's study the equation in the space $H^1([0, +\infty))$.

Theorem 5 *: if $\forall i = 1, ..., k$, $\tau_i(x) \in H^1([0,+\infty)) \cap L^4([0,+\infty))$, $\sqrt{x}\tau_i(x) \in L^2([0,+\infty))$, $\sqrt{x}\tau_i'(x) \in L^2([0,+\infty))$ and $r_0 \in H^1([0,+\infty))$, then the equation (4) has an unique solution in H^1, given by (12), which is a gaussian process with functional mean (8) and functional covariance (9).*

Proof: for the equation to have sense, we require that $\tau \in H^1$, $c \in H^1$, $\int_0^t \mathrm{Tr}\,(S_t Q S_u^*)\,du < +\infty\ \forall t$ and that A is the infinitesimal generator of a C^0 semigroup in H^1. For a proof of this last fact, we send the interested reader to the appendix. Now, let's check the second condition. As in the previous case, for $c \in H^1$ we give only a sufficient condition, which we will find necessary to have an invariant measure. We start seeing that $c \in L^2$:

$$\int_0^{+\infty} \tau^2(x) \left(\int_0^x \tau(u)\,du\right)^2 dx = \int_0^{+\infty} \tau^2(x) x \int_0^x \tau(u)^2\,du\,dx \le \|\tau\|_{L^2} \int_0^{+\infty} \tau^2(x) x\,dx$$

so a sufficient condition for c to be in L^2 is that $\sqrt{u}\tau(u) \in L^2$. Now we have to verify that the weak derivative of c:

$$\frac{\partial}{\partial x} c(x) = \frac{\partial \tau}{\partial x}(x) \int_0^x \tau(u)\,du + \tau^2(x)$$

is well defined. This is true, since $\tau \in L^2$. Now, let's verify that $c' \in L^2$:

$$\begin{aligned}
\|c'\|_{L^2}^2 &\le 2\left\|\frac{\partial \tau}{\partial x}\int_0^{\cdot} \tau(u)\,du\right\|_{L^2}^2 + 2\|\tau^2\|_{L^2}^2 = 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\int_0^x \tau(u)\,du\right)^2 dx + 2\|\tau\|_{L^4}^4 = \\
&= 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 \left(\int_0^x \tau(u)\,du\right)^2 dx + 2\|\tau\|_{L^4}^4 \le \\
&\le 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 x \int_0^x \tau^2(u)\,du\,dx + 2\|\tau\|_{L^4}^4 \le \\
&\le 2\int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 \int_0^{+\infty} \tau^2(u)\,du\;x\,dx + 2\|\tau\|_{L^4}^4 \le \\
&\le 2\|\tau\|_{L^2}^2 \int_0^{+\infty}\left(\frac{\partial \tau}{\partial x}(x)\right)^2 x\,dx + 2\|\tau\|_{L^4}^4 = 2\|\tau\|_{L^2}^2 \|\sqrt{x}\tau'(x)\|_{L^2}^2 + 2\|\tau\|_{L^4}^4
\end{aligned}$$

and so sufficient conditions for $c \in H^1$ are $\tau \in L^4$, $\sqrt{x}\tau(x) \in L^2$, $\sqrt{x}\tau'(x) \in L^2$.

The domain of A this time is the space:

$$D(A) = H^2([0,+\infty)) = W^{2,2}([0,+\infty)) = \{u \in L^2 \mid u' \in L^2, u'' \in L^2\}$$

where u' is the first weak derivative, and u'' the second weak derivative. One can verify, as before, that $(S_t)_t$ is a C^0 semigroup having infinitesimal generator A.

Finally:

$$\begin{aligned}
\int_0^t \mathrm{Tr}\,(S_t Q S_t^*)\,dt &= \int_0^t \|\tau(.+u)\|_{H^1}^2\,du \le \int_0^{+\infty} \|\tau(.+u)\|_{H^1}^2\,du = \\
&= \int_0^{+\infty}\int_0^{+\infty} (\tau^2(x+u) + \tau'^2(x+u))\,dx\,du =
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_x^{+\infty} (\tau^2(u)+\tau'^2(u))\ du\ dx \leq \\
&\leq \int_0^{+\infty} (\tau^2(u)+\tau'^2(u)) \int_0^u\ dx\ du = \\
&= \int_0^{+\infty} u(\tau^2(u)+\tau'^2(u))\ du = \|\sqrt{x}\tau(x)\|_{L^2} + \|\sqrt{x}\tau'(x)\|_{L^2}
\end{aligned}
$$

so the conditions $\sqrt{x}\tau(x)\in L^2$, $\sqrt{x}\tau'(x)\in L^2$ are sufficient for the integral to converge.

□

Now, let's check the conditions (1) and (3) of the proposition (2). As before, we have that:

$$Q = \tau(.)\otimes\tau(.)$$

$$Q_t = \int_0^t \tau(.+u)\otimes\tau(.+u)\ du$$

$$
\begin{aligned}
\sup_{t\geq 0} \mathrm{Tr}\ Q_t &= \int_0^{+\infty} \|\tau(.+u)\|^2_{H^1}\ du \leq \\
&\leq \|\sqrt{x}\tau(x)\|_{L^2} + \|\sqrt{x}\tau'(x)\|_{L^2}
\end{aligned}
$$

so the conditions $\sqrt{x}\tau(x)\in L^2, \sqrt{x}\tau'(x)\in L^2$ are necessary to have an invariant measure.

Let's check now the point (3). Let's look for a $b\in D(A)$ such that $Ab=-c$; then the condition is:

$$b(x) = b_0 - \int_0^x \tau(u)\int_0^u \tau(v)\ dv\ du = b_0 - \frac{1}{2}\left(\int_0^x \tau(u)\ du\right)^2$$

b is a decreasing function; then $b\in L^2$ only if $\lim_{x\to+\infty} b(x) = 0$. This determines b_0:

$$\lim_{x\to+\infty} b(x) = b_0 + \lim_{x\to+\infty}\frac{1}{2}\left(\int_0^x \tau(u)\ du\right)^2$$

the necessary conditions for this limit to be equal to 0 are that $\tau\in L^1$ and that $b_0 = -\frac{1}{2}\left(\int_0^{+\infty}\tau(u)\ du\right)^2$.

Now let's see under which condition $b\in L^2$:

$$
\begin{aligned}
\int_0^{+\infty} b^2(x)\ dx &= \frac{1}{4}\int_0^{+\infty}\left(\left(\int_0^{+\infty}\tau(u)\ du\right)^2 - \left(\int_0^x \tau(u)\ du\right)^2\right)^2\ dx = \\
&= \frac{1}{4}\int_0^{+\infty}\left(\int_x^{+\infty}\tau(u)\ du\right)^2\left(\int_0^x \tau(u)\ du + \int_0^{+\infty}\tau(u)\ du\right)^2 dx = \\
&= \frac{1}{4}\int_0^{+\infty}\left(\int_x^{+\infty}\tau(u)\ du\right)^2\left(\int_0^x \tau(u)\ du\right)^2\ dx +
\end{aligned}
$$

$$+\frac{1}{2}\|\tau\|_{L^1}^2 \int_0^{+\infty} \left(\int_x^{+\infty} \tau(u)\ du\right)^2 \left(\int_0^x \tau(u)\ du\right)\ dx$$

$$+\frac{1}{4}\|\tau\|_{L^1}^2 \int_0^{+\infty} \left(\int_x^{+\infty} \tau(u)\ du\right)^2\ dx \leq$$

$$\leq\ \|\tau\|_{L^1}^2 \int_0^{+\infty} \left(\int_x^{+\infty} \tau(u)\ du\right)^2\ dx$$

From the last but one passage we obtain that the condition $\int_x^{+\infty} \tau(u)\ du \in L^2$ is necessary to have $b \in L^2$; then, from the last passage, this condition is also sufficient.

Now we must verify that $b \in H^2$; for this it is sufficient that b' is well defined and belongs to H^1; now we remember that $Ab = b' = -c \in H^1$; then we found the b we were looking for.

Then, if the above condition is verified, every invariant measure is of the kind:

$$\mu = \nu * \delta_b * N(0, Q_\infty)$$

where $b \in D(A)$ is such that $Ab = -c$, and ν is an invariant measure for the equation:

$$dZ_t = (AZ_t)\ dt \tag{14}$$

then $b(x) = -z(x)$ is the unique function verifying the condition on b. Now, let's search for invariant measures for the equation (14). First of all let's search for solutions having as a marginal law a Dirac delta; this is equivalent to solve the differential equation:

$$\begin{cases} z' = Az \\ z_0 \in L^2 \end{cases}$$

in the space H^1, whose solution is $S_t z_0$, that is: $z(x,t) = z_0(x+t)$. Since we want a stationary solution, we must have that $S_t z_0 = z_0$, that is $z_0 = \text{const.}$; but the only constant function in H^1 is the zero, and so the only invariant measure of the kind δ_z is δ_0.

We claim that this is the only invariant measure for (14). To this aim we define, as usual, the following function classes:

$$B_b(H) = \{\phi : H \to \mathbf{R} \mid \phi \text{ measurable and bounded}\}$$

$$C_b(H) = \{\phi : H \to \mathbf{R} \mid \phi \text{ continuous and bounded}\}$$

$$M_1^+(H) = \{\mu \mid \mu \text{ probability measure on } (H, \mathcal{B}(H))\}$$

and the following operators:

$$P_t : B_b(H) \to B_b(H) \text{ such that } \forall\phi \in B_b(H), \forall x \in H : (P_t\phi)(x) = E[\phi(Z_t(x))]$$

$P : [0,T] \times H \times \mathcal{B}(H) \to \mathbf{R}$ such that $\forall t \in [0,T], x \in H, B \in \mathcal{B}(H)$:

$$P(t,x,B) = P_t \mathbf{1}_B(x) = E[\mathbf{1}_B(Z_t(x))] = \mathcal{L}(Z_t(x))(B)$$

$$P_t^* : M_1^+ \to M_1^+ \text{ such that } \forall \mu \in M_1^+, \forall B \in \mathcal{B}(H) : (P_t^* \mu)(B) = \int_H P(t,x,B)\ d\mu(x)$$

at last, we define the duality pairing between $B_b(H)$ and $M_1^+(\mathcal{B}(H))$:

$$< \phi, \mu >= \int_H \phi(x)\ d\mu(x)$$

Then we have that $\forall \phi \in B_b(H), \forall \mu \in M_1^+(H)$:

$$< P_t \phi, \mu >=< \phi, P_t^* \mu >$$

It is well known that, if we call $Z_t(X)$ the solution of the equation (14) at time t having initial data X and we call the law of X $\mathcal{L}(X) = \nu$, then $P_t^* \nu = \mathcal{L}(Z_t(X))$, and $P_t^* \delta_x = \mathcal{L}(Z_t(x))$ $\forall x \in H$. In our case, we have that if $f \in H^1$, then $P_t^* \delta_f = \delta_{S_t f}$. However, since the existence of $\nu \in M_1^+$ such that $\forall x \in H :\ P_t^* \delta_x \to^{weak} \nu$ implies that $\forall \mu \in M_1^+ :$ $P_t^* \mu \to^{weak} \nu$, then ν is the unique invariant measure, and, if H is an Hilbert space and $f_t \to f$ strongly, then $\delta_{f_t} \to^{weak} \delta_f$, then, since $\forall f \in H^1, \lim_{t \to +\infty} S_t f = 0$, all this implies that $P_t^* \delta_f \to^{weak} \delta_0$ $\forall f \in H^1$, and so that δ_0 is the only invariant measure for the equation (14).

We have just proved the following:

Theorem 6 *: given the equation (4) in the Hilbert space H^1, necessary and sufficient conditions to have an invariant measure are:*

$$\forall i = 1, ..., k : \begin{cases} \tau_i(x) \in H^1([0,+\infty)) \cap L^1([0,+\infty)) \\ \sqrt{x}\tau_i(x) \in L^2([0,+\infty)), \qquad \sqrt{x}\tau_i'(x) \in L^2([0,+\infty)) \\ \int_x^{+\infty} \tau_i(u)\ du \in L^2([0,+\infty)) \end{cases}$$

in these hypotheses only the following invariant measure exists:

$$\delta_{b^* + b_0} * \delta_0 * N(0, Q_\infty) = N(b^* + b_0, Q_\infty)$$

where:

$$b^*(x) + b_0 = \sum_{i=1}^k \int_0^x \tau_i(u) \int_0^u \tau_i(v)\ dv\ du - \int_0^{+\infty} \tau_i(u) \int_0^u \tau_i(v)\ dv\ du$$

$$Q_\infty = \int_0^{+\infty} \tau_i(. + u) \otimes \tau_i(. + u)\ du$$

□

7 Other two examples

Let's cite, only out of a mathematical curiosity, the same results as before in the case that the Hilbert spaces are:

$$H = L^2_\gamma([0,+\infty)) = \left\{ u : [0,+\infty) \to \mathbf{R} \;\middle|\; u \text{ meas. and s.t. } \int_0^{+\infty} u^2(x) e^{-\gamma x}\, dx < +\infty \right\}$$

$$H = L^2([0,+\infty)) = \left\{ u : [0,+\infty) \to \mathbf{R} \;\middle|\; u \text{ meas. and s.t. } \int_0^{+\infty} u^2(x)\, dx < +\infty \right\}$$

Theorem 7 *: if $\tau_i \in L^2_\gamma \cap L^2$ $\forall i = 1, ..., k$, and $r_0 \in L^2_\gamma$, then it exists an unique mild solution of the equation (4) in the space L^2_γ, given by (12). The solution is a gaussian process with functional mean (8) and covariance (9).*

□

Theorem 8 *: given the equation (4) in the Hilbert space L^2_γ, a necessary and sufficient condition in order to have an invariant measure is:*

$$\tau_i(x) \in L^2_\gamma([0,+\infty)) \cap L^2([0,+\infty)) \ \forall i = 1, ..., k$$

and there exist infinitely many invariant measures; in particular, the measures of the kind:

$$\delta_{z^*+c} * N(0, Q_\infty), c \in \mathbf{R}$$

$$\mu_f * N(0, Q_\infty), f \in L^2_\gamma \text{ periodical}$$

are invariant, where:

$$Q_\infty = \sum_{i=1}^{k} \int_0^{+\infty} \tau_i(.+u) \otimes \tau_i(.+u)\ du$$

$$z^*(x) = -\sum_{i=1}^{k} \int_0^{x} \tau_i(u) \int_0^{u} \tau_i(v)\ dv\ du$$

$f \in L^2_\gamma$ is periodical with period T and μ_f is the measure induced by the uniform density on $[0,T]$ via the application $t \to f(t+.)$.

□

Theorem 9 *: if $\tau_i(x) \in L^2([0,+\infty)), \sqrt{x}\tau_i(x) \in L^2([0,+\infty))$ $\forall i = 1, ..., k$ and $r_0 \in L^2([0,+\infty))$, then the equation (4) has a unique mild solution in L^2, given by (12), which is a gaussian process with functional mean (8) and functional covariance (9).*

$\square$

Theorem 10 *: given the equation (4) in the Hilbert space L^2, necessary and sufficient conditions in order to have an invariant measure are:*

$$\forall i = 1, ..., k : \begin{cases} \tau_i(x) \in L^2([0, +\infty)) \cap L^1([0, +\infty)) \\ \sqrt{x}\tau_i(x) \in L^2([0, +\infty)) \\ \int_x^{+\infty} \tau_i(u)\ du \in L^2([0, +\infty)) \end{cases}$$

in these hypotheses it exists the only invariant measure:

$$\delta_{z^*+z_0} * \delta_0 * N(0, Q_\infty) = N(z^* + z_0, Q_\infty)$$

where:

$$z^*(x) + z_0 = \sum_{i=1}^{k} \int_0^x \tau_i(u) \int_0^u \tau_i(v)\ dv\ du - \int_0^{+\infty} \int_0^{\infty} \tau_i(u) \int_0^u \tau_i(v)\ dv\ du$$

$$Q_\infty = \sum_{i=1}^{k} \int_0^{+\infty} \tau_i(. + u) \otimes \tau_i(. + u)\ du$$

$\square$

The proofs of these four theorems are analogous to those for the two Sobolev spaces we have seen before (see [7]).

8 The forward rate equation in the space $W^{1,1}_{loc}$

Now we try to solve the Musiela equation in the space of the absolutely continuous functions, which, as we have already seen, is its natural framework; but now, we notice that $W^{1,1}_{loc}$ is not an Hilbert space. This doen't allow us to use the invariant measure characterization theorem we used before. Besides, we must use a stochastic integration theory more general than the one used until now. In particular, we will embed the space $W^{1,1}_{loc}([0, +\infty))$ in the space of the Schwartz distributions on $\mathbf{R}$, which we will call $\mathcal{D}^*(\mathbf{R})$, about which a stochastic integration theory exists (see for example [5]), and we will analyse our equation in this space.

First of all we embed the space $W^{1,1}_{loc}$ in the distributions space via the canonical embedding which sends a function $f \in W^{1,1}_{loc}$ in the distribution $\phi \to \int_{-\infty}^{+\infty} \phi(x)f(x)\ dx$. This embedding is closed and continuous. Besides, we will indicate the standard duality pairing in this way: if $\phi \in C_0^\infty$, and $f \in W^{1,1}_{loc}$, then:

$$< \phi, f >= \int_{-\infty}^{+\infty} \phi(x)f(x)\ dx$$

We start verifying that (12) is solution of the equation also in the space $\mathcal{D}^*(\mathbf{R})$. To this aim we notice that $\mathcal{D}^*(\mathbf{R})$ is a locally convex space. This enables us to speak about locally equicontinuous C^0 semigroups of linear continuous operators in $\mathcal{D}^*(\mathbf{R})$, which have nice properties suitable for our aims. Besides this, we can talk about the infinitesimal generator A of a locally equicontinuous C^0 semigroup $(S_t)_t$ and of its domain $D(A)$ (see appendix). Finally, we notice that the process $(\sum_{i=1}^k \tau_i W_t^i)_t$ is a Brownian motion (in the sense specified in [5]) in $\mathcal{D}^*$, having functional mean 0 and functional covariance:

$$\begin{aligned} Cov(< f, \sum_{i=1}^k \tau_i W_s^i >, < g, \sum_{i=1}^k \tau_i W_t^i >) &= \sum_{ij=1}^k < f, \tau_i >< g, \tau_j > Cov(W_s^i, W_t^j) = \\ &= t \sum_{i=1}^k < f, \tau_i >< g, \tau_i > \ \ \forall f, g \in C_0^\infty(\mathbf{R}) \end{aligned}$$

Henceforth we use $(W_t)_t$ to indicate a generic Brownian motion in $\mathcal{D}^*$. Again we are interested in finding mild solutions:

Definition 11 *: we say that $(X_t)_t$ is a* **mild solution** *of the equation (4) if $\forall f \in \mathcal{D}$:*

$$< f, X_t >=< f, X_0 > + \int_0^t (< A^* f, X_u > + < f, c >) \ du + \int_0^t < f, dW_u >$$

We start with a general theorem about the Langevin equation in $\mathcal{D}^*(-\infty, +\infty)$.

Theorem 12 *: if A^* generates a locally equicontinuous semigroup $(S_t)_t$ in $\mathcal{D}$, $(W_t)_t$ is a generic Brownian motion in $\mathcal{D}^*$, and X_0 is $\mathcal{F}_0$-measurable, then the process $(X_t)_t$, defined by:*

$$X_t = S_t X_0 + \int_0^t S_{t-u} c \ du + \int_0^t S_{t-u} \ dW_u \tag{15}$$

is the only mild solution to the Langevin equation:

$$\begin{cases} dX_t = (AX_t + c) \ dt + dW_t \\ X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; \mathcal{D}^*) \end{cases} \tag{16}$$

Besides, if X_0 is gaussian, then $(X_t)_t$ is a gaussian process with functional mean

$$E[< f, r_t >] =< f, S_t E[r_0] > + \int_0^t < f, S_{t-u} c > \ du \tag{17}$$

and functional covariance:

$$Cov(< f, r_t >, < g, r_v >) = Cov(< f, S_t r_0 >< g, S_v r_0 >) + \int_0^t p(S_{t-u}^* f, S_{v-u}^* g) \ du \tag{18}$$

Proof: first of all we notice that, since $(S_t)_t$ is a locally equicontinuous semigroup, then for all continuous seminorm p on $\mathcal{D}^*(\mathbf{R})$ and $\forall T > 0$, there exist a continuous seminorm q on $\mathcal{D}^*(\mathbf{R})$ such that $p(S_t f) \leq q(f)$ $\forall t \in [0, T], f \in \mathcal{D}^*(\mathbf{R})$, so $\int_0^t p(S_{t-u} c) \ du \leq tq(c)$, and so the first integral in (15) has a sense; besides there exists r such that $q <_{HS} r$, and so $\int_0^t \|S_{t-u}\|_{HS(p,r)}\ dt \leq \int_0^t (t-u)(q :_{HS} r)\ dt < +\infty$ and so the stochastic integral is well defined in the sense of [5].

Now we prove that (15) is a solution. For this aim, we define the process:

$$Y_t = X_t - S_t X_0 - \int S_{t-s} c\ ds = \int_0^t S_{t-u}\ dW_u$$

then $Y_0 = 0$, and $(Y_t)_t$ has the stochastic differential:

$$\begin{aligned}
dY_t &= dX_t - AS_t X_0\ dt - c\ dt - \left(\int_0^t AS_{t-s} c\ ds \right)\ dt = \\
&= (AX_t + c)\ dt + dW_t - AS_t X_0\ dt - c\ dt + A \int_0^t S_{t-s} c\ ds = \\
&= AX_t\ dt + dW_t - AS_t X_0\ dt + A \int_0^t S_{t-s} c\ ds = \\
&= AY_t\ dt + dW_t
\end{aligned}$$

and so $(Y_t)_t$ satisfies the equation:

$$\begin{cases} dY_t = AY_t\ dt + dW_t \\ Y_0 = 0 \end{cases} \tag{19}$$

if and only if $(X_t)_t$ satisfies (16).

We notice that the stochastic integral in the expression is the integral of a deterministic functional $\mathcal{D}^* \to \mathcal{D}^*$, so we can write it in this way:

$$\begin{aligned}
\int_0^t S_{t-u}\ dW_u &= [S_{t-u} W_u]_0^t + \int_0^t AS_{t-u} W_u\ du = \\
&= W_t + \int_0^t AS_{t-u} W_u\ du
\end{aligned}$$

Then we can write:

$$< f, Y_t >=< f, W_t > + \int_0^t < f, AS_{t-u} W_u >\ du$$

Using the formula:

$$f(t) - f(0) = \int_0^t \frac{d}{dv} f(v)\ dv$$

which is valid if $f : [0,T] \to W^{1,1}_{loc}$ is differentiable, we arrive to:

$$\begin{aligned}
< f, Y_t > &= \int_0^t < f, dW_u > + \int_0^t \left(< A^* f, W_v > + \int_0^v < A^{*2} S^*_{v-u} f, W_u >\ du \right)\ dv = \\
&= \int_0^t < f, dW_u > + \int_0^t \left(< f, W_v > + \int_0^v < A^* S^*_{v-u} f, W_u >\ du \right)\ dv = \\
&= \int_0^t < f, dW_u > + \int_0^t < A^* f, Y_u >\ du
\end{aligned}$$

so the process $(Y_t)_t$ is a solution of the equation (19) and the process $(X_t)_t$ is a solution of the equation (16).

Now, let's prove the uniqueness. For this aim, we need the following lemma:

Lemma 13 *: if $(Y_t)_t$ is a mild solution of (19), then $\forall \psi \in C^1([0,T], D(A^*)), t \in [0,T]$ we have:*

$$< \psi(t), Y_t >= \int_0^t < \psi'(s) + A^*\psi(s) >, Y_s > \ ds + \int_0^t < \psi(s), dW_s >$$

Proof: let's consider a function $\psi(t) = \psi_0.\phi(t)$, where $\psi_0 \in D(A^*), \phi \in C^1([0,T], \mathbf{R})$; then, if we set:

$$F(t) =< \psi_0, Y_t >= \int_0^t < A^*\psi_0, Y_s > \ ds+ < \psi_0, W_t >$$

we have:

$$\begin{aligned} d[F(t)\phi(t)] &= \phi(t) \ dF(t) + F(t)\phi'(t) \ dt = \\ &= \phi(t) < A^*\psi_0, Y_t > \ dt + \phi(t) < \psi_0, dW_t > +\phi'(t) < \psi_0, Y_t > \ dt \end{aligned}$$

and so:

$$\begin{aligned} F(t)\phi(t) &= \int_0^t \phi(s) < A^*\psi_0, Y_s > \ ds + \int_0^t \phi(s) < \psi_0, dW_s > + \int_0^t \phi'(s) < \psi_0, Y_s > \ ds = \\ &= \int_0^t < A^*\psi(s), Y_s > \ ds + \int_0^t < \psi'(s), Y_s > \ ds + \int_0^t < \psi(s), dW_s > \end{aligned}$$

and so the lemma holds for all the functions of the kind $\psi(t) = \psi_0.\phi(t)$; since this class is dense in $C^1([0,T], D(A^*))$, the lemma is proved

□

To prove the theorem, we put $\psi(t) = S^*_{t-s}\phi_0$, with $\phi_0 \in D(A^*)$; then:

$$< S_{t-t} \ \phi_0 \ , Y_t >=< \phi_0, Y_t >= \int_0^t < S^*_{t-s}\phi_0, dW_s >= \int_0^t < \phi_0, S_{t-s} \ dW_s >$$

then, since $D(A^*)$ is dense in $\mathcal{D}$, $< \phi_0, Y_t >= \int_0^t < \phi_0, S_{t-s} \ dW_s > \ \forall \phi_0 \in \mathcal{D}$, and so $Y_t = \int_0^t S_{t-s} \ dW_s$ can be the only solution to (19).

Now, let's verify that, if X_0 is gaussian, the solution (15) is a gaussian process. In order to do this, we only need to check that $\forall t_1, ..., t_n$ the random variable $(X_{t_1}, ..., X_{t_n})$ is gaussian in $(\mathcal{D}^*)^n$. This means that $\forall \phi_1, ..., \phi_n \in (C_0^\infty)^n$ the real random variable $(< \phi_1, X_{t_1} >, ..., < \phi_n, X_{t_n} >)$ has to be gaussian. This is equivalent to check that $\forall \phi_1, ..., \phi_n \in \mathcal{D}^*$:

$$\begin{aligned} \sum_1^n i < \phi_i, X_{t_i} > &= \sum_1^n i < \phi_i, S_{t_i} X_0 > + \sum_1^n i < \phi_i, \int_0^{t_i} S_{t_i - u} c \ du > + \\ &+ \sum_1^n i < \phi_i, \int_0^{t_i} S_{t_i - u} \ dW_u > \end{aligned}$$

the first addend is a real gaussian random variable, and the second one is a real number. Let's verify that the third one is a real gaussian random variable:

$$\begin{aligned}\sum_1^n i < \phi_i, \int_0^{t_i} S_{t_i-u}\ dW_u > &= \sum_1^n i < \phi_i, \int_0^T \mathbf{1}_{[0,t_i)} S_{t_i-u}\ dW_u >= \\ &= \sum_1^n i < \int_0^T \mathbf{1}_{[0,t_i)} S^*_{t_i-u}\phi_i,\ dW_u >= \\ &= < \int_0^T \sum_1^n i \mathbf{1}_{[0,t_i)} S^*_{t_i-u}\phi_i,\ dW_u >\end{aligned}$$

which is the stochastic integral of a deterministic linear operator from $\mathcal{D}$ into itself, and so it is a real gaussian random variable. The law of the process is then uniquely determined by his functional mean and covariance.

Now we only need to prove that the process has functional mean (17) and functional covariance (18). If we have $f, g \in C_0^\infty$, then, for the properties of the stochastic integral (see [5]):

$$\begin{aligned}Cov(< f, r_t > &, < g, r_v >) = \\ &= Cov\left(< f, S_t r_0 + \int_0^t S_{t-u}\ dW_u >, < g, S_v r_0 + \int_0^v S_{v-u}\ dW_u >\right) = \\ &= Cov(< f, S_t r_0 >< g, S_v r_0 >) + \\ &\quad + E\left[< f, \int_0^t S_{t-u}\ dW_u >< g, \int_0^t S_{v-u}\ dW_u >\right] = \\ &= Cov(< f, S_t r_0 >< g, S_v r_0 >) + \int_0^t p(S^*_{t-u} f, S^*_{v-u} g)\ du\end{aligned}$$

□

Now we apply this general theorem to our special case.

Corollary 14 *: if $\tau_i \in W^{1,2}_{loc}(\mathbf{R}^+)$ $\forall i = 1, ..., k$ and $r_0 \in W^{1,1}_{loc}(\mathbf{R}^+)$ $\mathbf{P}$-a.s., then the equation (4) has an unique solution $(r_t)_t$ in $\mathcal{D}^*(\mathbf{R})$, with $r_t \in W^{1,1}_{loc}(\mathbf{R})$ $\forall t$ $\mathbf{P}$-a.s., given by (12), which is a gaussian process with functional mean (8) and functional covariance (9).*

Proof: first of all we have to verify that our translation semigroup $(S_t)_t$ is an equicontinuous C^0 semigroup having as infinitesimal generator A; we send the interested reader to the appendix; then we notice that

$$c(x) = \int_0^x \tau(u) \int_0^u \tau(v)\ dv\ du$$

is an element of $W^{1,1}_{loc}$. In fact c has weak derivative equal to:

$$\frac{\partial}{\partial x} c(x) = \frac{\partial \tau}{\partial x}(x) \int_0^x \tau(u)\ du + \tau^2(x)$$

which is well defined. Now we must verify that $c \in L^1_{loc}$, and $c' \in L^1_{loc}$; $\forall T \in (0, +\infty)$ we have:

$$\int_0^T |c(x)|\ dx = \int_0^T \tau(x) \int_0^x \tau(u)\ du\ dx = \frac{1}{2}\left(\int_0^T \tau(x)\ dx\right)^2$$

$$\begin{aligned}
\int_0^T |c'(x)|\ dx &\leq \int_0^T |\tau'(x)| \int_0^x \tau(u)\ du\ dx + \int_0^T \tau^2(x)\ dx = \\
&= \left[\int_0^x |\tau'(u)|\ du \int_0^x \tau(u)\ du\right]_0^T - \int_0^T \left(\int_0^x |\tau'(u)|\ du\right) \tau(x)\ dx + \\
&\quad + \|\tau\|^2_{L^2([0,T])} \leq 2\|\tau'\|_{L^1([0,T])}\|\tau\|_{L^1([0,T])} - \|\tau\|^2_{L^2([0,T])}
\end{aligned}$$

Since $\tau \in W^{1,1}_{loc}$, it follows that τ is continuous, and $\tau \in L^2_{loc}$. Then $c \in W^{1,1}_{loc}$. Now, since all the hypotheses of the theorem (12) hold, it follows that (12) is solution of (4).

We finally have to check that $r_t \in W^{1,1}_{loc}(\mathbf{R}^+)$ $\mathbf{P}$-a.s. If we look at the explicit solution (12), this is true, because S_t sends $W^{1,1}_{loc}(\mathbf{R}^+)$ in itself, so the Bochner integral (second addend) is an integral in $W^{1,1}_{loc}(\mathbf{R}^+)$, and the stochastic integral (third addend) takes values in $W^{1,2}_{loc}(\mathbf{R}^+)$ $\mathbf{P}$-a.s.; in fact, $\forall T > 0$, $W^{1,2}([0,T])$ is a Hilbert space, and the definition of stochastic integral in $W^{1,2}([0,T])$ is equivalent to the one on $\mathcal{D}^*$. Besides we have:

$$\begin{aligned}
E\left[\left\|\int_0^t S_{t-u}\tau\ dW_u\right\|^2_{W^{1,2}([0,T])}\right] &= \int_0^t \mathrm{Tr}\ S_{t-u}QS^*_{t-u}\ dt = \int_0^t \|S_{t-u}\tau\|^2_{W^{1,2}([0,T])}\ dt = \\
&= \int_0^t \int_0^T (\tau^2(x+t-u) + \tau'^2(x+t-u))\ dx
\end{aligned}$$

if $\tau \in W^{1,2}([0, T+t])$, then the last integral converges, and so $\| \int_0^t S_{t-u}\tau\ dW_u\|^2_{W^{1,2}([0,T])}$ is a.s. finite $\forall T > 0$. So, if $\tau \in W^{1,2}_{loc}(\mathbf{R}^+)$, then $\int_0^{\cdot} S_{\cdot-u}\tau\ dW_u \in W^{1,2}_{loc}(\mathbf{R}^+)$ $\mathbf{P}$-a.s.

Now we only need to prove that the process (12) has functional mean (8). Now let's verify that the functional covariance is really (9). If we have $f, g \in C^\infty_0$, then, since in our case $p(f,g) =< f, \tau >< g, \tau >$:

$$\begin{aligned}
Cov(< f, r_t > \quad &, \quad < g, r_v >) = Cov(< f, S_t r_0 >< g, S_v r_0 >) + \int_0^t p(S^*_{t-u}f, S^*_{v-u}g)\ du = \\
&= Cov(< f, S_t r_0 >< g, S_v r_0 >) + \int_0^t < S^*_{t-u}f, \tau >< S^*_{v-u}g, \tau >\ du = \\
&= E[< f, S_t r_0 >< g, S_v r_0 >] + \\
&\quad + \int_0^t < f(.), \tau(. + t - u) > . < g(.), \tau(. + v - u) >\ du
\end{aligned}$$

For the covariance we will use this notation:

$$Cov(r_t, r_v) = S_v Cov(r_0, r_0) S^*_t + \int_0^t \tau(. + t - u) \otimes \tau(. + v - u)\ du$$

□

Now we search for an invariant law following this idea. Since, if we have a gaussian initial data, the solution process will be gaussian, and since in this case the stationarity is equivalent to the weak stationarity (that is, only the functional mean and covariance are stationary), we only need to find an initial data for which the solution process is stationary to prove that there is a gaussian invariant measure.

For this aim, we notice that we already have some natural candidates to be invariant measures, that is the gaussian ones on H^1_γ. Then let's check if and under what conditions the gaussian invariant measures on H^1_γ are still invariant measures in $W^{1,1}_{loc}$. The general gaussian invariant measure was $N(b^*(.)+b_0, Q_\infty)$, where:

$$b^*(x) = -\int_0^x \tau(u) \int_0^u \tau(v)\ dv\ du$$

$$Q_\infty = \int_0^{+\infty} \tau(.+u) \otimes \tau(.+u)\ du$$

and b_0 was a generic real number.

First of all we prove that a gaussian measure of the kind $N(b^*, Q_\infty)$ exists on $\mathcal{D}^*(\mathbf{R})$; in particular, we construct it such a way that it is the distribution of a $\mathcal{D}^*(\mathbf{R})$-valued random variable r_0, and $\mathbf{P}\{r_0 \in W^{1,2}_{loc}(\mathbf{R}^+)\} = N(b^*, Q_\infty)(W^{1,2}_{loc}(\mathbf{R}^+)) = 1$. In ordere to do this, we take a real k-dimensional brownian motion $(Z_t)_t$, independent from $(W_t)_t$ and a real constant b_0 and consider the random variable:

$$r_0 = b_0 + \int_0^{+\infty} S_u c\ du + \sum_{i=1}^k \int_0^{+\infty} S_u \tau_i\ dZ^i_u$$

Proposition 15 *: if $\tau_i \in L^1 \cap L^2 \cap W^{1,2}_{loc}(\mathbf{R}^+)$ $\forall i = 1, ..., k$, then r_0 has distribution $N(b^*, Q_\infty)$ and $\mathbf{P}\{r_0 \in W^{1,1}_{loc}(\mathbf{R}^+)\} = 1$.*

Proof: since r_0 is the limit of gaussian random variables:

$$r_0 = \lim_{t \to +\infty} \left(b_0 + \int_0^t S_u c\ du + \int_0^t S_u \tau\ dZ_u \right)$$

then, if the limit converges in $\mathcal{D}^*$, then r_0 is well defined and gaussian. In order to check the convergence of r_0, we only need to check that $E[r_0]$ and $Cov(r_0, r_0)$ converge.

If $\tau \in L^1$, then:

$$\begin{aligned} E[r_0] - b_0 &= \int_0^{+\infty} S_u c\ du = \int_0^{+\infty} \tau(x+u) \int_0^{x+u} \tau(v)\ dv\ du = \\ &= \int_x^{+\infty} \tau(u) \int_0^u \tau(v)\ dv\ du = \int_0^{+\infty} \tau(u) \int_0^u \tau(v)\ dv\ du + \\ &\quad - \int_0^x \tau(u) \int_0^u \tau(v)\ dv\ du = \frac{1}{2} \left(\int_0^{+\infty} \tau(u)\ du \right)^2 - \frac{1}{2} \left(\int_0^x \tau(u)\ du \right)^2 \end{aligned}$$

then $E[r_0]$ is well defined and belongs to $W^{1,1}_{loc}$. Besides, if $\tau \in L^2$, and we take $f \in C_0^\infty$ and suppose that $[a,b] = supp(f)$, then we have:

$$\begin{aligned}
Cov \quad & (< f, r_0 >, < f, r_0 >) = E\left[\lim_{t\to+\infty} < f, \int_0^t S_u\tau\ dZ_u >^2\right] = \\
&= \lim_{t\to+\infty} E\left[< f, \int_0^t S_u\tau\ dZ_u >^2\right] = \lim_{t\to+\infty} \int_0^t < f, S_u\tau >^2\ du = \\
&= \int_0^{+\infty} \left(\int_0^{+\infty} f(x)\tau(x+t)\ dx\right)^2\ dt \le \|f\|_\infty \int_0^{+\infty} \left(\int_a^b \tau(x+t)\ dx\right)^2\ dt \le \\
&\le \|f\|_\infty \int_0^{+\infty} (b-a) \int_a^b \tau^2(x+t)\ dt\ dx = \|f\|_\infty (b-a) \int_0^{+\infty} \int_{a+t}^{b+t} \tau^2(u)\ du\ dt = \\
&= \|f\|_\infty (b-a) \int_a^{+\infty} \tau^2(u) \int_{(u-b)\vee 0}^{u-a}\ dt\ du \le \\
&\le \|f\|_\infty (b-a) \int_a^{+\infty} \tau^2(u)(u-a-u+b)\ du = \|f\|_\infty (b-a)^2 \|\tau\|^2_{L^2}
\end{aligned}$$

since $|Cov(< f, r_0 >, < g, r_0 >)| \le E[< f, r_0 >^2]E[< g, r_0 >^2]\ \forall f, g \in C_0^\infty$, then we have that if $\tau \in L^2$, then $Cov(< f, r_0 >, < g, r_0 >)$ is a real number.

Now we only have to prove that $\mathbf{P}\{r_0 \in W^{1,1}_{loc}(\mathbf{R}^+)\} = 1$. If we look at the definition of r_0, this is true, because S_t sends $W^{1,1}_{loc}(\mathbf{R}^+)$ in itself, so the Bochner integral (second addend) is an integral in $W^{1,1}_{loc}(\mathbf{R}^+)$. Moreover, the stochastic integral (third addend) takes values in $W^{1,2}_{loc}(\mathbf{R}^+)$ $\mathbf{P}$-a.s.; in fact, $\forall T > 0$, $W^{1,2}([0,T])$ is a Hilbert space and the definition of stochastic integral in $W^{1,2}([0,T])$ is equivalent to the one in $\mathcal{D}^*$. Besides we have:

$$\begin{aligned}
E \quad & \left[\lim_{t\to+\infty} \left\|\int_0^t S_u\tau\ dZ_u\right\|^2_{W^{1,2}([0,T])}\right] = \lim_{t\to+\infty} \int_0^t \mathrm{Tr}_{\ W^{1,2}([0,T])} S_u Q S_u^*\ du = \\
&= \int_0^{+\infty} \|S_u\tau\|^2_{W^{1,2}([0,T])}\ du = \int_0^{+\infty} \int_0^T (\tau^2(x+u) + \tau'^2(x+u))\ dx\ du = \\
&= \int_0^T \int_0^{+\infty} (\tau^2(x+u) + \tau'^2(x+u))\ du\ dx \le T\|\tau\|_{W^{1,2}(\mathbf{R}^+)}
\end{aligned}$$

If $\tau \in W^{1,2}(\mathbf{R}^+)$, then the last integral converges, and so $\| \int_0^t S_u\tau\ dW_u\|^2_{W^{1,2}([0,T])}$ is a.s. finite $\forall T > 0$, and so $r_0 \in W^{1,1}_{loc}$ $\mathbf{P}$-a.s.

□

Now, let's try to start with an initial data with law $N(b^*(x) + b_0, Q_\infty)$, and see what is the law of the solution process. Since the law is gaussian, we only need to calculate the functional mean and covariance. Let's calculate the mean:

$$\begin{aligned}
E[r_t](x) &= E[S_t r_0](x) + \int_0^t (S_{t-u}c)(x)\ du = \\
&= -\int_0^{x+t} \tau(u) \int_0^u \tau(v)\ dv\ du + b_0 + \int_0^t \tau(x+t-u) \int_0^{x+t-u} \tau(v)\ dv\ du =
\end{aligned}$$

$$
\begin{aligned}
&= b_0 - \int_0^{x+t} \tau(u) \int_0^u \tau(v)\ dv\ du + \int_x^{x+t} \tau(u) \int_0^u \tau(v)\ dv\ du = \\
&= b_0 - \int_0^{x} \tau(u) \int_0^u \tau(v)\ dv\ du = b_0 + b(x)
\end{aligned}
$$

Now, let's calculate the functional covariance of the process. If $f, g \in C_0^\infty$, then:

$$
\begin{aligned}
Cov(< f, r_t > \quad , \quad & < g, r_{t+h} >) = E[< f, S_t r_0 >< g, S_{t+h} r_0 >] + \\
& + \int_0^t < f(.), \tau(. + t - u) > . < g(.), \tau(. + t + h - u) > \ du = \\
= \ & E[< S_t^* f, r_0 >< S_{t+h}^* g, r_0 >] + \\
& + \int_0^t \int_0^{+\infty} f(x)\tau(x + t - u)\ dx \int_0^{+\infty} g(y)\tau(y + t + h - u)\ dy\ du = \\
= \ & \int_0^{+\infty} \int_0^{+\infty} \tau(x + u + t) f(x)\ dx \int_0^{+\infty} \tau(y + u + t + h) g(y)\ dy\ du + \\
& + \int_0^t \int_0^{+\infty} f(x)\tau(x + u)\ dx \int_0^{+\infty} g(y)\tau(y + h + h)\ dy\ du = \\
= \ & \int_t^{+\infty} \int_0^{+\infty} \tau(x + u) f(x)\ dx \int_0^{+\infty} \tau(y + u + h) g(y)\ dy\ du + \\
& + \int_0^t \int_0^{+\infty} f(x)\tau(x + u)\ dx \int_0^{+\infty} g(y)\tau(y + h + h)\ dy\ du = \\
= \ & \int_0^{+\infty} \int_0^{+\infty} \tau(x + u) f(x)\ dx \int_0^{+\infty} \tau(y + u + h) g(y)\ dy\ du = \\
= \ & \int_0^{+\infty} \int_0^{+\infty} \tau(x + u) f(x)\ dx \int_h^{+\infty} \tau(y + u) g(y - h)\ dy\ du = \\
= \ & E[< f, r_0 >< S_h^* g, r_0 >] = \\
= \ & E[< f, r_0 >< g, S_h r_0 >] = Cov(< f, r_0 >, < g, r_h >)
\end{aligned}
$$

As in the space H_γ^1, also here we found infinitely many invariant measures. This non uniqueness of the invariant measure is caused by the fact that the constant functions, as in the case H_γ^1, belong to $W_{loc}^{1,1}$.

We can collect the results we just found in the following:

Theorem 16 *: if $\tau_i \in W_{loc}^{1,2} \cap L^1 \cap L^2$ $\forall i = 1, ..., k$, then the equation (4) has infinitely many invariant measures in $W_{loc}^{1,1}$. In particular, the measures of the kind $N(b^*(.) + b_0, Q_\infty)$ are the only gaussian invariant measures.*

□

9 Appendix

In this appendix we collect some proof of results we used before.

First of all we recall the definition of C^0 semigroup in a Banach space. Let's take a real Banach space E, endowed with the norm $\|.\|_E$. We call $L(E,E)$ the linear space of the linear continuous operators from E in itself; this is a Banach space too, endowed with the norm:

$$\|A\|_{L(E,E)} = \sup\{\|Ax\|_E \mid x \in E, \|x\| = 1\}$$

We say that $\{S_t\}_{t\geq 0}$ is a C^0 **semigroup of linear continuous operators**, or more simply a C^0**semigroup**, if it has the following properties:

1) $S_t \in L(H,H)\ \forall t \geq 0$
2) $S_0 = I$
3) $S_{t+s} = S_t \circ S_s$

4) $\forall x \in H : \lim_{t\to 0^+} S_t x = x$

Now we define the following linear subspace of E:

$$D = D(A) = \left\{ x \in E \;\middle|\; \exists \lim_{t\to 0^+} \frac{S_t x - x}{t} \right\}$$

and the linear operator A on $D(A)$:

$$\forall x \in D(A) : Ax = \lim_{t\to 0^+} \frac{S_t x - x}{t}$$

D is called **domain** of A, and A is called **infinitesimal generator** of the semigroup $\{S_t\}_{t\geq 0}$. We recall that $D(A)$ is dense in E, and $\forall x \in D(A) : \frac{d}{dt} S_t x = AS_t x = S_t Ax$.
For a proof, see [8].

Now that we did these conventions, let's prove the results we used. We pose:

$$A = \frac{\partial}{\partial x}$$
$$S_t f(x) = f(x+t)$$

Lemma 17 *: $(S_t)_t$ is a C^0 semigroup in the Hilbert space H^1_γ, having as infinitesimal generator A, which has domain equal to H^2_γ.*

Proof: first of all we must check the semigroup conditions $(1), ..., (4)$:
1) $S_t \in L(H^1_\gamma, H^1_\gamma) = L$ e $\|S_t\|_L \leq e^{\frac{\gamma}{2}u}$; in fact, if $f \in H^1_\gamma$, then:

$$\begin{aligned}
\|S_t f(x)\|_{H^1_\gamma} &\leq \sqrt{\int_0^{+\infty} f^2(x+t)\gamma e^{-\gamma x}\, dx} + \sqrt{\int_0^{+\infty} f'^2(x+t)\gamma e^{-\gamma x}\, dx} = \\
&= e^{\frac{\gamma t}{2}} \left(\sqrt{\int_0^{+\infty} f^2(x+t)\gamma e^{-\gamma(x+t)}\, dx} + \sqrt{\int_0^{+\infty} f'^2(x+t)\gamma e^{-\gamma(x+t)}\, dx} \right) =
\end{aligned}$$

$$= e^{\frac{\gamma t}{2}} \left(\sqrt{\int_t^{+\infty} f^2(x)\gamma e^{-\gamma x}\ dx} + \sqrt{\int_t^{+\infty} f'^2(x)\gamma e^{-\gamma x}\ dx} \right) \leq e^{\frac{\gamma t}{2}} \|f\|_{H^1_\gamma}$$

so $\|S_t\|_L \leq e^{\frac{\gamma t}{2}}$.

2-3) trivial verification

4) if $f \in C^1([0,+\infty))$, then:

$$\lim_{t\to 0} \|S_t f - f\|_{H^1_\gamma} = \lim_{t\to 0} \gamma \int_0^{+\infty} ((f(t+x) - f(x))^2 + (f'(t+x) - f(x))^2) e^{-\gamma x}\ dx = 0$$

then, since C^1 is dense in H^1_γ, we have that $\forall \varepsilon, \forall g \in H^1_\gamma\ \exists f \in C^1$ such that $\|f-g\|_{H^1_\gamma} < \varepsilon$; then:

$$\begin{aligned} \|S_t g - g\|_{H^1_\gamma} &\leq \|S_t g - S_t f\|_{H^1_\gamma} + \|S_t f - f\|_{H^1_\gamma} + \|f-g\|_{H^1_\gamma} \leq \\ &\leq \|S_t\|_L \|f-g\|_{H^1_\gamma} + \varepsilon + \|f-g\|_{H^1_\gamma} \leq (e^{\gamma t} + 2)\varepsilon \end{aligned}$$

Now let's prove that A is the infinitesimal generator of $(S_t)_t$; if $f \in H^2_\gamma = W^{2,2}_\gamma$, then $f \in C^1$, and we have:

$$\lim_{t\to 0^+} \frac{S_t f - f}{t}(x) = \lim_{t\to 0^+} \frac{f(x+t) - f(x)}{t} = \frac{\partial f}{\partial x}$$

which is well defined and belongs to H^1_γ; so A is the infinitesimal generator of $(S_t)_t$, and $H^2_\gamma \subseteq D(A)$; conversely, $f' \in H^1_\gamma$ only if $f \in H^2_\gamma$, so $D(A) = H^2_\gamma$.

$\square$

Lemma 18 *: $(S_t)_t$ is a C^0 semigroup in H^1, having as infinitesimal generator A, which has the domain H^2.*

Proof: we have to check the semigroup conditions $(1), ..., (4)$:

1) $S_t \in L(H^1, H^1) = L$ e $\|S_t\|_L \leq 1$; in fact if $f \in H^1$, then:

$$\begin{aligned} \|S_t f(x)\|_{H^1} &\leq \sqrt{\int_0^{+\infty} f^2(x+t)\ dx} + \sqrt{\int_0^{+\infty} f'^2(x+t)\ dx} = \\ &= \sqrt{\int_t^{+\infty} f^2(x)\ dx} + \sqrt{\int_t^{+\infty} f'^2(x)\ dx} \leq \|f\|_{H^1} \end{aligned}$$

so $\|S_t\|_L \leq 1$.

2-3) trivial verification

4) if $f \in C^1([0,+\infty))$, then:

$$\lim_{t\to 0} \|S_t f - f\|_{H^1} = \lim_{t\to 0} \int_0^{+\infty} ((f(t+x) - f(x))^2 + (f'(t+x) - f(x))^2)\ dx = 0$$

then, since C^1 is dense in H^1, we have that $\forall \varepsilon > 0, \forall g \in H^1 \; \exists f \in C^1$ such that $\|f - g\|_{H^1} < \varepsilon$; then:

$$\begin{aligned} \|S_t g - g\|_{H^1} &\leq \|S_t g - S_t f\|_{H^1} + \|S_t f - f\|_{H^1} + \|f - g\|_{H^1} \leq \\ &\leq \|S_t\|_L \|f - g\|_{H^1} + \varepsilon + \|f - g\|_{H^1} \leq 3\varepsilon \end{aligned}$$

Now, let's prove that A is the infinitesimal generator of $(S_t)_t$; if $f \in D(A) = W^{2,2}$, then $f \in C^1$, and we have:

$$\lim_{t \to 0^+} \frac{S_t f - f}{t}(x) = \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} = \frac{\partial f}{\partial x}$$

which is well defined and belongs to H^1; so A is the infinitesimal generator of $(S_t)_t$, and $H^2 \subseteq D(A)$; conversely, $f' \in H^1$ only if $f \in H^2$, so $D(A) = H^2$.

$\square$

For the space $\mathcal{D}(\mathbf{R})$ we need a different notion, and we also need to know his structure. First of all we notice that we can represent $\mathbf{R}$ in this way:

$$\mathbf{R} = \cup_{n \in \mathbf{Z}} I_n^o, \qquad \text{where } I^o = \text{ interior of } I$$

where $I_n = [n-1, n+1] \; \forall n \in \mathbf{Z}$; let $\{\alpha_n\}_n \subseteq \mathcal{D}$ a partition of unity corresponding to the covering $\{I_n^o | n \in \mathbf{Z}\}$, and such that $\alpha_n(x) = \alpha(x+n) \; \forall n \in \mathbf{Z}, x \in \mathbf{R}$ and $\alpha_n > 0$ in I_n^o; then for every $\phi \in \mathcal{D}$, $\alpha_n \phi$ is a C^∞ function with support in I_n.

Now, for all I_n, we define $\mathcal{D}(I_n)$ to be the space consisting of all functions on a compact interval $[a, b]$ that can be extended to C^∞ functions on $\mathbf{R}$ vanishing outside of I_n. Then $\mathcal{D}(I_n)$ is a dense vector subspace of $L^2(I_n)$.

We consider the following operator:

$$D = -\frac{d^2}{dx^2} : \mathcal{D} \to L^2$$

We denote the closed extension of D by the same notation D. The functions:

$$s_n(x) = \sin nx$$

form an orthonormal base in L^2. We have that:

$$D s_n = n^2 s_n$$

We define the following norms on $\mathcal{D}(I_n)$:

$$\|f\|_p = \|D^p f\|_{L^2(I_n)}, \qquad p \in \mathbf{N}$$

then we notice that:

$$\|f\|_p^2 = \sum_{n=1}^{\infty} n^{4p} < f, s_n >_{L^2}^2 \tag{20}$$

We also define the seminorms on $\mathcal{D}(\mathbf{R})$:

$$\|f\|_{n,p} = \|\alpha_n f\|_p \ \forall n \in \mathbf{N}, p \in \mathbf{N}$$

Then both $\|.\|_p$ and $\|.\|_{n,p}$ are Hilbertian seminorms, i.e. they have the following property:

$$p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2$$

and we have that $\|.\|_p \leq_{HS} \|.\|_{p+1}$, and so $\|.\|_{n,p} \leq_{HS} \|.\|_{n,p+1}$.

If we now take two sequences $(a_n)_n$, $(p_n)_n$ such that $a_n > 0$ and $p_n \in \mathbf{N}$, we can define the norm:

$$\|f\|_{(a_n),(p_n)} = \sum_{n \in \mathbf{N}} a_n^2 \|f\|_{n,p_n}^2$$

Such a norm is always well defined, because the support of $f \in \mathcal{D}$ intersects only a finite number of the intervals $(I_n)_n$, and so it turns out to be a Hilbertian norm. We can now endow the space $\mathcal{D}$ with the multi-Hilbertian topology τ defined by $(\|\cdot\|_{(a_n),(p_n)})$, i.e. the weakest topology in which these norms are continuous, and we find out that it coincides with the Schwartz topology, which is given by the seminorms:

$$|f|_{p,K} = \max_{n \leq p} \sup_{x \in K} |f^n(x)|, \qquad K \subseteq \mathbf{R} \text{ compact}$$

We obtain that:

$$\|\cdot\|_{(a_n),(p_n)} \leq_{HS} \|\cdot\|_{(b_n),(p_n+1)}, \qquad b_n = n a_n (\|.\|_{n,p_n} : \|.\|_{n,p_n+1})_{HS}$$

If we define the **Kolmogorov I-topology** of τ (denoted by $I(\tau)$) the weakest topology defined by the seminorms:

$$\{p | \exists q \leq \tau \text{ such that } p \leq_{HS} q\}$$

then τ is **nuclear**, that is $\tau = I(\tau)$.

From all this discussion, it follows that $\mathcal{D}$ is a locally convex space, and we denote by $\mathcal{D}^*$ his dual, known as the **Schwartz distributions** space. For a more detailed discussion of these facts, see [5].

For the space $\mathcal{D}$, we need the following notion: we say that $(S_t)_t$ is an **equicontinuous C^0 semigroup of linear continuous operators**, or more simply a **equicontinuous C^0 semigroup** in the locally convex space X, if it has the following properties:

1) $S_t \in L(X, X) \ \forall t \geq 0$
2) $S_0 = I$
3) $S_{t+s} = S_t \circ S_s$
4) the family $(S_t)_t$ is **equicontinuous** in t, that is, for every continuous seminorm p on X it exists a continuous seminorm q on X such that $p(S_t x) \leq q(x) \ \forall t \geq 0, x \in X$.

Also in this case we can define the **domain** of A:

$$D(A) = \left\{ x \in X \,\middle|\, \exists \lim_{t \to 0^+} \frac{S_t x - x}{t} \right\}$$

and call **infinitesimal generator** of $(S_t)_t$ the linear operator A defined as follows:

$$\forall x \in D(A) : Ax = \lim_{t \to 0^+} \frac{S_t x - x}{t}$$

We remind that also in this case $D(A)$ is dense in X, and: $\forall x \in D(A) : \frac{d}{dt} S_t x = A S_t x = S_t A x$.

For a proof of these facts, see [8].

For some applications, we can find the condition (4) too strong to verify. We can give the following definition: we say that $(S_t)_t$ is a **locally equicontinuous** C^0 **semigroup**, in the locally convex space X, if it has the properties (1), (2), (3) and:

4') the family $(S_t)_t$ is **locally equicontinuous** in t, that is, $\forall T \in \mathbf{R}^+$ and for every continuous seminorm p on X it exists a continuous seminorm q_T on X such that $p(S_t x) \leq q_T(x)$ $\forall t \in [0, T], x \in X$.

if we define $D(A)$ and A as before, then we have no conditions about $D(A)$, but A enjoy the same properties seen before. We start with the

Lemma 19 *if* $t \in [\frac{1}{2}, \frac{1}{2}]$*, and if we call* $b = \max_{x \in [-1,1]} \alpha(x)$ *and* $a = \min_{x \in [-\frac{1}{2}, \frac{1}{2}]} \alpha(x)$*, then:*

$$\|S_t f\|_{0,0}^2 \leq \frac{b}{a} (\|f\|_{-1,0}^2 + \|f\|_{0,0}^2 + \|f\|_{1,0}^2) \ \forall f \in \mathcal{D}$$

Proof: first of all we define the function:

$$I_n(x) = a.\mathbf{1}_{[n-\frac{1}{2}, n+\frac{1}{2}]}(x), \quad n \in \mathbf{N}$$

then:

$$\begin{aligned} \|S_t f\|_{0,0}^2 &= \|\alpha_0 S_t f\|_{L^2}^2 \leq \|b S_t f\|_{L^2}^2 \leq \frac{b}{a} (\|I_{-1} f\|_{L^2}^2 + \|I_0 f\|_{L^2}^2 + \|I_1 f\|_{L^2}^2) \leq \\ &\leq \frac{b}{a} (\|\alpha_{-1} f\|_{L^2}^2 + \|\alpha_0 f\|_{L^2}^2 + \|\alpha_1 f\|_{L^2}^2) = \frac{b}{a} (\|f\|_{-1,0}^2 + \|f\|_{0,0}^2 + \|f\|_{1,0}^2) \end{aligned}$$

$\square$

Corollary 20 *if* $t \in [\frac{1}{2}, \frac{1}{2}]$*, then* $\forall p \in \mathbf{N}$*:*

$$\|S_t f\|_{0,p}^2 \leq \frac{b}{a} (\|f\|_{-1,p}^2 + \|f\|_{0,p}^2 + \|f\|_{1,p}^2) \ \forall f \in \mathcal{D}$$

Proof: it follows immediately from (20) and from the fact that:

$$< f, g >_{L^2} = \frac{1}{4} (\|f + g\|_{L^2}^2 + \|f - g\|_{L^2}^2)$$

□

Proposition 21 *: if we call $(S_t^*)_t$ the adjoint semigroup of $(S_t)_t$, that is:*

$$S_t^* f(x) = f(x-t)\ \forall f \in C_0^\infty(\mathbf{R}) = \mathcal{D}(\mathbf{R})$$

and A^* the adjoint of A, that is:

$$A^* = -\frac{\partial}{\partial x}$$

then $(S_t)_t$ is a locally equicontinuous C^0 semigroup in $\mathcal{D}(\mathbf{R})$, having as infinitesimal generator A^*, which has the domain all $\mathcal{D}(\mathbf{R})$.

Proof: we must check the semigroup conditions $(1), ..., (3)$ and $(4')$:

1) $S_t^* \in L(\mathcal{D}(\mathbf{R}), \mathcal{D}(\mathbf{R}))$; in fact, for every seminorm p continuous in $\mathcal{D}(\mathbf{R})$ it exists a norm $\|.\|_{(a_n),(p_n)}$ and $\beta \in \mathbf{R}^+$ such that:

$$p(f) \leq \beta \|f\|_{(a_n),(p_n)}\ \forall f \in \mathcal{D}(\mathbf{R})$$

Now, we'll check that $\forall t > 0$, $\exists \|.\|_{(b_n),(q_n)}$ such that $\|S_t f\|_{(a_n),(p_n)} \leq \|f\|_{(b_n),(q_n)}\ \forall f \in \mathcal{D}$.

We can suppose (after a suitable shifting of n) that $t \in [-\frac{1}{2}, \frac{1}{2}]$; then we define:

$$b_n^2 = \frac{b}{a}(a_{n-1}^2 + a_n^2 + a_{n+1}^2)$$
$$q_n = \max\{p_{n-1}, p_n, p_{n+1}\}$$

then we have:

$$\|S_t^* f\|_{(a_n),(p_n)} \leq \|f\|_{(b_n),(q_n)}\ \forall f \in \mathcal{D}(\mathbf{R})$$

in fact:

$$\begin{aligned}
\|S_t^* f\|_{(a_n),(p_n)} &= \sum_{n\in\mathbf{Z}} a_n^2 \|S_t^* f\|_{n,p_n}^2 \leq \sum_{n\in\mathbf{Z}} a_n^2 \frac{b}{a} \|f\|_{n-1,p_n}^2 + \|f\|_{n,p_n}^2 + \|f\|_{n+1,p_n}^2) \leq \\
&\leq \sum_{n\in\mathbf{Z}} b_n^2 \|f\|_{n,q_n}^2 = \|f\|_{(b_n),(q_n)}^2
\end{aligned}$$

and so $S_t^* \in L(\mathcal{D}(\mathbf{R}), \mathcal{D}(\mathbf{R}))\ \forall t$

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4') if we define:

$$b_n^2 = \frac{b}{a}(a_{n-1}^2 + a_n^2 + ... + a_{n+1+[T]}^2)$$
$$q_n = \max\{p_{n-1}, p_n, ..., p_{n+1+[T]}\}$$

then we have:

$$\|S_t^* f\|_{(a_n),(p_n)} \leq \|f\|_{(b_n),(q_n)} \; \forall f \in \mathcal{D}(\mathbf{R}), t \in [0,T]$$

The proof is as point (1); so, $(S_t^*)_t$ is locally equicontinuous in t.

Now let's prove that A^* is the infinitesimal generator of $(S_t^*)_t$; if $f \in \mathcal{D}(\mathbf{R})$, then $f \in C^1$, and we have:

$$\lim_{t \to 0^+} \frac{S_t^* f - f}{t}(x) = \lim_{t \to 0^+} \frac{f(x-t) - f(x)}{t} = \lim_{t \to 0^-} \frac{f(x+t) - f(x)}{-t} = -\frac{\partial f}{\partial x}$$

which is well defined and belongs to $\mathcal{D}(\mathbf{R})$; so A^* is the infinitesimal generator of $(S_t^*)_t$, and $D(A) = \mathcal{D}(\mathbf{R})$.

□

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