Formule ricorsive per variabili aleatorie binomiali e di Poisson

A volte capita di dover calcolare probabilità binomiali o di Poisson "vicine":

$$\mathbb{P}\{X \leq K\} = \sum_{0}^{K} {}_{\ell}\, \mathbb{P}\{X = \ell\}$$

Questo calcolo può essere reso più efficiente se si riesce ad usare ricorsivamente, per $\mathbb{P}\{X=\ell\}$, le probabilità già calcolate.

<u>formule ricorsive per v.al. binomiali</u>

Se $X \sim B(n,p)$, allora

$$\mathbb{P}\{X = \ell+1\} = \binom{n}{\ell+1} p^{\ell+1}(1-p)^{n-\ell-1} =$$

$$= \frac{n!}{(\ell+1)!\,(n-\ell-1)!}\, p^{\ell+1}(1-p)^{n-\ell-1}$$

$$\mathbb{P}\{X = \ell\} = \frac{n!}{\ell!\,(n-\ell)!}\, p^{\ell}(1-p)^{n-\ell}$$

Allora:

$$\frac{\mathbb{P}\{X=\ell+1\}}{\mathbb{P}\{X=\ell\}} = \frac{\frac{\cancel{n!}}{(\ell+1)\cancel{!}\,\cancel{(n-\ell-1)!}}\, \cancel{p^{\ell+1}}\cancel{(1-p)^{n-\ell-1}}}{\frac{\cancel{n!}}{\cancel{\ell!}\,(n-\ell)\cancel{!}}\, \cancel{p^{\ell}}(1-p)^{\cancel{n-\ell}}} =$$

$$\overline{\cancel{l!}\,(n-l)\cancel{!}}$$

$$= \frac{n-l}{l+1}\,\frac{p}{1-p}$$

Infine:

$$\mathbb{P}\{X = l+1\} = \mathbb{P}\{X = l\}\,\frac{n-l}{l+1}\cdot\frac{p}{1-p}$$

<u>esempio</u> (es. 2.7 riv.)

$$S_6 \sim \mathcal{B}\left(6\,;\,\frac{1}{2}\right)$$

$$\mathbb{P}\{S_6 = 0\} = \underbrace{\binom{6}{0}}_{=1}\left(1-\frac{1}{2}\right)^6 = 0{,}015625$$

$$\mathbb{P}\{S_6 = 1\} = \mathbb{P}\{S_6 = 0\}\times\frac{6}{1}\times\frac{1/2}{1/2} = 0{,}015625\times 6 = 0{,}09375$$

$$\mathbb{P}\{S_6 = 2\} = \mathbb{P}\{S_6 = 1\}\times\frac{5}{2} = 0{,}09375\cdot\frac{5}{2} = 0{,}234375$$

$$\mathbb{P}\{S_6 = 3\} = \mathbb{P}\{S_6 = 2\}\times\frac{4}{3} = 0{,}234375\cdot\frac{4}{3} = 0{,}3125$$

$$\mathbb{P}\{S_6 = 4\} = \mathbb{P}\{S_6 = 3\}\times\frac{3}{4} = 0{,}3125\cdot\frac{3}{4} = 0{,}234375$$

$$\mathbb{P}\{S_6 = 5\} = \mathbb{P}\{S_6 = 4\}\cdot\frac{2}{5} = 0{,}234375\cdot\frac{2}{5} = 0{,}09375$$

$$\mathbb{P}\{S_6 = 6\} = \mathbb{P}\{S_6 = 5\}\cdot\frac{1}{6} = 0{,}09375\cdot\frac{1}{6} = 0{,}015625$$

<u>formula ricorsiva per v. al. di Poisson</u>

formula ricorsiva per v. al. di Poisson

Se $X \sim Po(\lambda)$, allora

$$\frac{\mathbb{P}\{X = K+1\}}{\mathbb{P}\{X = K\}} = \frac{\cancel{e^{-\lambda}} \frac{\lambda^{\cancel{K+1}}}{(K+1)\cancel{!}}}{\cancel{e^{-\lambda}} \frac{\cancel{\lambda^{K}}}{\cancel{K!}}} = \frac{\lambda}{K+1}$$

quindi

$$\mathbb{P}\{X = K+1\} = \mathbb{P}\{X = K\} \cdot \frac{\lambda}{K+1}$$

esercizio : $X \sim Po(2)$

$$\mathbb{P}\{X \leq 3\} = \sum_{k=0}^{3} \mathbb{P}\{X = K\}$$

$$\mathbb{P}\{X = 0\} = e^{-2} \frac{2^0}{0!} = e^{-2} = 0,135$$

$$\mathbb{P}\{X = 1\} = e^{-2} \cdot \frac{2^1}{1!} = 0,135 \cdot \frac{2}{1} = 0,27$$

reg. ric. $= 0,135 \times \frac{2}{1} = 0,27$

$$\mathbb{P}\{X = 2\} = \mathbb{P}\{X = 1\} \frac{\lambda}{2} = 0,27 \times \frac{2}{2} = 0,27$$

$$\mathbb{P}\{X = 3\} = \mathbb{P}\{X = 2\} \frac{\lambda}{3} = 0,27 \cdot \frac{2}{3} = 0,18$$

$$\mathbb{P}\{X \leq 3\} = 0,135 + 0,27 + 0,27 + 0,18 = 0,855$$