On the superreplication approach for European interest rates derivatives

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Abstract

In this paper we analyse the superreplication approach to stochastic volatility in the case of European interest rates derivatives. We exploit some general results of [13] and [17] to prove that the minimal superstraegy is given by the solution of a nonlinear PDE associated to the model, that is the so-called Black-Scholes-Barenblatt (BSB) equation. In particular we show how this approach apply to the case of cap and floor extending results of [6].

1 Introduction

In this paper we analyse the superreplication approach to stochastic volatility in the case of European interest rates derivatives including also cases when the value function is non-smooth. More precisely our starting point are some general results of [17] about the characterization of superstrategy for European multiasset derivatives via the regular ($C^{1,2}$) solutions of the associated BSB equation and of [13] about existence, uniqueness, regularity of the solution of the Black-Scholes-Barenblatt (BSB) equation and characterization of superstrategies in the case when the solution is not $C^{1,2}$. We show that such results can be applied in the case of interest rates derivatives with suitable arrangements. In particular we treat the case of caps and floors (that do not fit directly into the case of European derivatives due to the presence of multiple maturities) and we show that our approach gives, in the Gaussian case, an extension of a result of [6].

We give now an outline of the problem with a formulation that uses forward prices (see e.g. [7]). Such outline can be found also in the quoted papers, but we prefer to repeat it here for the reader's convenience. We have a riskless asset, whose value we suppose constant through time, and n risky assets whose prices $S = (S^1, \ldots, S^n)$ follow the dynamic

$$dS_t = \bar{S}_t \sigma_t \, dW_t,$$

where W is a n-dimensional Brownian motion under the so called forward-neutral measure $\mathbb{Q}$, the matrix process σ is adapted and takes values in a closed bounded set $\Sigma \subset M(n, n, \mathbb{R})$, and $\bar{S}_t = \text{diag } (S_t)$. We then consider an agent who wants to price and hedge a European contingent claim whose payoff is a deterministic function $h(\cdot)$ which is globally Lipschitz continuous, calculated in S_T. Since the market could be incomplete because of the stochastic volatility σ and the agent is not able to hedge the volatility, he chooses to hedge the option by using the superhedging approach. Following e.g. [2], [14] [17], we assume that he fixes the price of the option as $C_t = C(t, S_t)$ and builds a self-financing portfolio consisting of a quantity $\Delta_t^i = \frac{\partial C}{\partial S_i}(t, S_t)$ of the asset $S^i, i = 1, \ldots, n$, where $C(t, S)$ is a solution of a nonlinear PDE, similar to the Black-Scholes equation, called Black-Scholes-Barenblatt (BSB) equation in analogy with [2] and [17]. This equation is a Hamilton-Jacobi-Bellman equation and it is linked to a stochastic control problem that have a nice financial interpretation (see section 2 and also [13, 17]). In fact, the agent could build a "subjective" model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \, dW_t$$

where the state variable S^γ has to be controlled by the "subjective" volatility γ in order to maximize the payoff function

$$\mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} h(S_T^\gamma) \mid \mathcal{F}_t \right]$$

corresponding to the no-arbitrage price of the contingent claim at time t. In this way, the agent wants to maximize in γ his payoff protecting himself against "the worst possible case", and the BSB equation is the Hamilton-Jacobi-Bellman equation corresponding to this optimal control problem.

The main known results on the relationship between the BSB equation and the superhedging of a European contingent claim are substantially Theorem 3 and Theorem 6 (recalled in Section 2 for the reader's convenience). Theorem 3 characterizes superstrategies in terms of the solutions of the BSB equation *when they are regular enough to apply Itô formula* (C^1 in time and C^2 in space). This result has been proved in the single asset case in [2] and in the multiasset case in [14] where a detailed discussion on the model and on its financial interpretation is given, (see also [17] for a simpler proof of it), and it is also shown (using known results on PDEs) that if the BSB equation is uniformly parabolic, then its solution is smooth enough to apply Itô formula. Uniform parabolicity of the BSB equation depends essentially from the set Σ being composed of invertible matrices. This is not verified in some financial examples (see Section 8 of [17] for a typical situation). For this reason it is interesting to see what can be said in non uniformly parabolic cases and Theorem 6 presented and proved in [13] furnish a partial answer in this case. We notice that many other papers on this subject and related ones have been written. We recall e.g. [9] where the problem was considered first and also [1, 2, 6, 8, 10, 12, 18]. We refer to [13] and [17] for a wider introduction to the problem and a more complete bibliography on it.

In this paper the results above are applied to the case of interest rates derivatives. We point out that such extension is not trivial since the setup for interest rates derivatives is in fact different and some basic assumptions of the general model of [13] and [17] are not satisfied (in particular for the case of caps and floors, where we have multiple maturities).

The paper is organized as follows: Section 2 is devoted to present the problem and to collect some known material (mainly taken from [13] and [17]) needed in the sequel. In Section 3 we present the main results on superreplication of interest rates derivatives with a single maturity (e.g. swaptions); in Section 4 we consider the case of caps and we show that our approach can characterize the superstrategies and, in the case of Gaussian interest rates models can allow to calculate explicitly such superstrategy.

2 The general model

This section is devoted to collect the material (mostly taken from [13] and [17]) concerning the general problem of superreplication of European multiasset derivatives via the BSB equation. It is divided in subsection as follows:

- the first one devoted to the general setting,
- the second one devoted to the BSB equation,
- the third one devoted to characterisation theorems for superstrategies,
- the fourth one containing a result of [17] that, in some case, allows to find explicit solution of BSB equation in some case and an example where it can be applied (which is exploited in section 4 to study the case of caps).

2.1 Setting of the model

We suppose that there exists a riskless asset M and n risky assets S^i, $i = 1, \dots, n$ in the market. We make the usual assumptions that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ complete and right continuous, where $\mathcal{F}_t$ represents the information available up to time t and that M and S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Besides, we assume that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is called **forward-neutral probability** [7], such that the value of the riskless asset M remains constantly equal to 1 through time, and the dynamics of the assets S^i under $\mathbb{Q}$ are the following:

$$dS_t^i = S_t^i \langle \sigma_t^i, dW_t \rangle$$

where $(W_t)_t$ is a d-dimensional $\mathbb{Q}$-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^d$ and σ^i is a d-dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\Sigma)$, where Σ is a closed bounded set in the space of $n \times d$ real matrices $M(n, d, \mathbb{R})$ and $\mathcal{A}(\Sigma)$ (which we call set of **admissible volatilities**) is the set of Σ-valued processes progressively measurable with respect to $(\mathcal{F}_t)_t$.

We can write the dynamics of the risky assets in a more compact vectorial notation in this way:

$$dS_t = \bar{S}_t \sigma_t \, dW_t$$

where we let $\bar{S}_t = \text{diag}\,(S_t)$.

Now we consider an operator in the market that wants to create and sell a European derivative asset with payoff $h(S_T) = h(S_T^1, \dots, S_T^n)$, where h is a locally Lipschitz continuous function having polinomial growth.

Remark 1 We do not assume that the market is complete. In particular, the filtration $(\mathcal{F}_t)_t$ could be strictly larger than the one generated by W or than the one generated by S. In this case we have a genuine stochastic volatility model and the market is incomplete. For this reason, we can assume that the number of assets n is different from the number of Brownian motions d in the model. Moreover, the assumption that the interest rate is zero can be achieved by a so-called change of numeraire, that is by expressing the prices of all the assets in the market in units of a zero coupon bond with maturity T (for more details, see [7]).

In order to price and hedge the asset, the agent fixes a price C_t for the asset at time t and builds a self-financing portfolio Π, holding η_t units of the money market account M and Δ_t^i units of the asset S^i at time t. We indicate with Π_t the value of the portfolio at time t, defined by:

$$\Pi_t = \eta_t + \langle \Delta_t, S_t \rangle \tag{1}$$

where $\Delta_t = (\Delta_t^1, \dots, \Delta_t^n)$. In order to prevent arbitrage opportunities, we say that the portfolio is **admissible** if Π is a supermartingale: in this way, if $\Pi_0 = 0$, we have $\mathbb{E}[\Pi_T] \leq 0$, so there are not arbitrage opportunities in the market. We say that the portfolio is **self-financing** if Π follows the dynamic

$$\begin{cases} d\Pi_t = \langle \Delta_t, dS_t \rangle \\ \Pi_0 = C_0 \end{cases} \tag{2}$$

If the portfolio is self-financing, the process Δ is sufficient to characterize it, and $\eta_t = \Pi_t - \langle \Delta_t, S_t \rangle$.

The market is not complete because of the stochastic volatility σ, so the agent chooses to hedge the claim using the superhedging approach. To this aim, we define the **tracking error**:

$$e_t = \Pi_t - C_t$$

so $e_0 = 0$ by definition of Π. Intuitively, the tracking error gives the error made by the operator in estimation, or better, the difference between the hedging portfolio held by the agent and the option sold. If the portfolio is admissible and self-financing, we say that (C, Δ) is a

- **superhedging strategy**, or simply **superstrategy** if e is a non decreasing process
- **subhedging strategy**, or simply **substrategy**, if e is a non increasing process
- **hedging strategy** if e is identically equal to zero.

If one builds a superstrategy, one can successfully hedge a short position in the contingent claim C_t, while if one builds a substrategy, one can successfully hedge a long position in C_t. In particular with a superstrategy we have that $e_T \geq 0$, that is $\Pi_T \geq C_T = h(S_T)$, so the portfolio succeeds in superhedging the contingent claim. We notice that superstrategies and substrategies are good candidates to be arbitrages: in fact if we have (say) $\Pi_T \geq C_T$ $\mathbb{Q}$-a.s. and $\Pi_T > C_T$ with positive probability, then the agent succeeds in making a profit with no initial endowment. This means that the value C_t in a superstrategy has to be interpreted as an arbitrage upper bound for the price of the claim at time t; similarly, the value C_t in a substrategy has to be interpreted as a lower bound. In fact, if the price of the claim is C_t or less, one could build an arbitrage by buying the claim and by selling short the subhedging portfolio for the same price, thus making the final profit $C_T - \Pi_T = -e_T \geq 0$.

Remark 2 The substrategy case is in some sense symmetric to the superstrategy one. In fact, if C defines a superstrategy for the claim defined by h, then $-C$ defines a substrategy for the claim defined by $-h$. This allows us to analyse only one of the two cases, and to have automatically results for the other. For this reason, in the rest of the paper we will analyse only the superstrategy case, and it will be implicit that analogous results will hold also for substrategies.

As already outlined by [2], [8], [13], [14], and [17], in order to have a superstrategy, a natural procedure is this: the agent fixes the price C_t of the option and the quantities Δ_t of the risky assets in the hedging portfolio Π at time t as

$$C_t = C(t, S_t), \qquad \Delta_t^i = \frac{\partial C}{\partial s_i}(t, S_t)$$

where C is the solution of the following partial differential equation:

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \operatorname{tr}\, (D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*) = 0, & t \in [0,T), s \in \mathbb{R}_+^n \\ C(T,s) = h(s), & s \in \mathbb{R}_+^n \end{cases} \tag{3}$$

where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s), \ldots, \frac{\partial C}{\partial s_n}(t,s))$, and $D^2C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (3), as in [2], will be called from now on the **Black-Scholes-Barenblatt (BSB) equation** for h. Moreover, by Remark 2, in order to have a substrategy it is sufficient to proceed as above by substituting the maximum operator in Equation (3) with a minimum.

The approach above has a stochastic control interpretation: we assume that the agent does not know the volatility σ, but in order to hedge the asset he can use the "subjective" model

$$S_t^\gamma = S_t^\sigma, \qquad S_\tau^\gamma = \bar{S}_\tau^\gamma \gamma_\tau \, dW_\tau, \qquad t < \tau < T. \tag{4}$$

Here the initial datum S_t^σ is the present market price of the assets at time t which depends of course on the real but unknown volatility σ up to time t. The process $\gamma \in \mathcal{A}_t(\Sigma)$ is another admissible volatility starting from time t, that we can interpret as a control corresponding to the subjective volatility of the agent used in order to decide his strategy. If the market were complete and the dynamics were given by Equation (4), then the arbitrage free price of the European contingent claim would be

$$C_t^{\mathbb{Q},\gamma} = \mathbb{E}_{\mathbb{Q}} \left[h(S_T^\gamma) \;\middle|\; S_t^\sigma \right] \tag{5}$$

where $\mathbb{Q}$ would be the unique forward-neutral measure of the problem. Since the market is not complete, the agent fixes the price of the option by maximising the quantity above both in $\mathbb{Q}$ as in γ, thus protecting himself against "the worst possible case". It turns out that the BSB equation is the Hamilton-Jacobi-Bellman equation corresponding to this optimal control problem. Under the assumptions of our model, the value function of the optimal control problem above, defined by

$$V_{\mathbb{Q}}^+(t,s) = \sup_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}} \left[h(S_T^\gamma) \middle| \, S_t^\gamma = s \right] \tag{6}$$

is the unique viscosity solution C of Equation (3) (see Theorem 4 in the following subsection) and does not depend on the particular choice of $\mathbb{Q}$. Thus we have that $C(t, S_t^\sigma)$ defines the upper bound of the admissible arbitrage free prices for the claim at time t, in the sense that if the price of the claim is greater than $C(t, S_t^\sigma)$, then it is possible to build an arbitrage in the market by selling the claim and buying the superreplicating portfolio. The same reasoning can be repeated for the substrategy.

2.2 The characterization theorems

We now present the characterization theorems for superstrategies. We indicate here with $C_p^{1,2}([0,T)\times\mathbb{R}_+^n)$ the space of functions C that are continuous and with polynomial growth on $[0,T)\times\mathbb{R}_+^n$ together with their first derivative in t and first and second derivatives in s.

Theorem 3 *If we restrict ourselves to the case $C_t = C(t,S_t)$, with $C\in C_p^{1,2}\left([0,T)\times\mathbb{R}_+^n\right)$, then:*

i) (C,Δ) is a superstrategy if and only if $\Delta_t = DC(t,S_t)$ and

$$R_t = -\frac{\partial C}{\partial t}(t,S_t) - \frac{1}{2}\mathrm{tr}\ D^2C(t,S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*$$

is non negative $\mathbb{Q}$-a.s. for all $t\in[0,T]$.

ii) If C is the solution of Equation (3) and $\Delta_t = DC(t,S_t)$, then (C,Δ) is a superstrategy. Moreover there is not any superstrategy (C',Δ') such that $C'_t < C(t,S_t^\gamma)$ with positive probability for some t and for all $\gamma\in\mathcal{A}(\Sigma)$.

This is a characterization theorem of the superstrategies. In fact, provided the value function $C\in C_p^{1,2}$, the theorem gives the minimal superstrategy, in the sense that there does not exist a cheaper strategy that succeeds in giving an increasing tracking error for all $\gamma\in\mathcal{A}(\Sigma)$. Since the agent does not know the true volatility σ, he/she must protect himself against all the possible volatilities $\gamma\in\mathcal{A}(\Sigma)$ and this is the best result he can expect.

An analogous of Theorem 2 holds for the case of a substrategy, provided we reverse the inequalities and substitute the max operator in Equation (3) with a min.

The final conclusion of Theorem 3 is that if a smooth solution of Equation (3) exists, then there exists a Markov superstrategy, and it is given by the space derivative of the solution of Equation (3). Conversely, if this does not happen we cannot apply Theorem 3 without proving new regularity results that are unknown at this stage. Then the characterization theorems for superstrategies strongly depends on the properties of solutions of the BSB equation, especially the regularity of them. For this reason we give then a brief look to the known results on this topic (see [13] for bibliography, statement and proofs, and [11] for a wide introduction to viscosity solutions).

The first step is to recall a general result of existence and uniqueness of viscosity solutions for the BSB equation (3) that always holds true in our case.

Theorem 4 *Let Σ be compact, h be locally Lipschitz continuous and h, Dh have polynomial growth. Then the value function $C=V^+$ defined by Equation (6) is a viscosity solution of Equation (3) in $[0,T]\times\mathbb{R}_+^n$. Moreover, C is the unique viscosity solution having polynomial growth that satisfies the boundary condition $C(T,s)=h(s)$ for all $s\in\mathbb{R}_+^n$.*

The second step is to prove smoothness of the viscosity solution. In order to do this we use results on uniformly parabolic equations. For sake of brevity we only give a sketch of the procedure without precise statements. The first thing is to make the change of variable $y_i = \log s_i$ in the BSB equation. Then the BSB equation transforms into a nonlinear PDE with constant coefficients which is uniformly parabolic if and only if $\det\gamma\gamma^*\neq 0$ for all $\gamma\in\Sigma$. If this condition is satisfied we can use results on uniformly parabolic equations of [4, 19] obtaining the following result.

Theorem 5 *Let Σ be compact, h be locally Lipschitz continuous and h, Dh have polynomial growth. If $\Sigma^2 \subseteq \mathrm{GL}(n,\mathbb{R})$ then the viscosity solution C of the BSB equation (3) belongs to $C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$.*

If the BSB equation (3) is not uniformly parabolic (i.e. if there exists $\gamma \in \Sigma$ such that $\det \gamma\gamma^* = 0$), a version of Theorem 3 can be proved under stronger assumptions. More precisely it can be proved that the process (C, Δ) defined of Section 2 gives a superstrategy when the final payoff h is a convex (or semiconvex) function of the assets and for all t the law of the random variable S_t is absolutely continuous with respect to the Lebesgue measure. This case does not cover all the possible payoffs (for example, it does not cover call-spreads), but indeed many of the multiasset European options (call options on the maximum, call and put options on the mean, exchange options, etc.) are covered.

We also observe that the analogous results for a substrategy needs to assume that h is concave, not convex and so the symmetry between sub- and superstrategies results here is broken. In fact, most of the examples require h to be convex (and generally not concave). So, if we want to get a result on existence of substrategies when h is convex we need the additional assumption that the set Σ be convex (see [13]).

We have the following result, proved in [13].

Theorem 6 *If the assumptions*

(i) The payoff h is convex (or semiconvex) and Lipschitz continuous and h, Dh have polynomial growth.

(ii) For every $t > 0$ the law of the random variable S_t^σ has a density with respect to the Lebesgue measure which is absolutely continuous.

hold and C denotes the unique viscosity solution of Eq. (3), then $\Delta_t = D_s C(t, S_t)$ is a.s. well defined and $(C_t, \Delta_t)_t$ is a superstrategy, where $C_t = C(t, S_t)$.

Remark 7 In fact all the results above could be generalized in the case when the set Σ of expected volatilities is time-varying. It is enough to ask a Hölder continuity of it with respect to time.

2.3 Explicit solutions via reduction to the Black and Scholes case

Now we quote some results from [17] that will be useful in the sequel.

We notice that Eq. (3) contains an optimization problem. The agent has to solve this problem if he wants to solve the BSB equation numerically, and if he wants to have a view of what could be the "worst case" against him. Moreover, it turns out that in some cases the problem gives a simple way to pass from the BSB equation to a Black-Scholes equation, which is simpler both from a theoretical point of view as well as from the numerical side. In any case, this problem is significant only if $C \in C^{1,2}$, so in the following we will make that assumption.

The optimization problem is

$$\max_{\gamma \in \Sigma} F_{t,s}(\gamma) \ , \tag{7}$$

where $F_{t,s} : \Sigma \to \mathbb{R}$ is defined by

$$F_{t,s}(\gamma) = \mathrm{tr}\ \left(\bar{s} D^2 C(t,s) \bar{s} \gamma\gamma^*\right) = \mathrm{tr}\ \left(A_{t,s}\gamma\gamma^*\right)\ , \tag{8}$$

where $A_{t,s} = \bar{s}D^2C(t,s)\bar{s}$ and C is the solution of Eq. (3). We will often write F and A instead of $F_{t,s}$ and $A_{t,s}$ if there is no risk of ambiguity.

An important case is when the optimization problem has a constant solution. In this case, we can reduce the non linear BSB equation to a standard Black-Scholes equation (see [3] as the following result (contained in [17]) states.

Proposition 8 *If $C \in C^{1,2}([0,T) \times \mathbb{R}^n)$ is a solution of the Black-Scholes equation*

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\mathrm{tr}\,(D^2C(t,s)(\bar{s}\bar{\gamma})(\bar{s}\bar{\gamma})^*) = 0, & t \in [0,T), s \in \mathbb{R}^n_+ \ , \\ C(T,s) = h(s), & s \in \mathbb{R}^n_+ \ , \end{cases} \tag{9}$$

with $\bar{\gamma} \in \Sigma$, such that the problem (7) attains its maximum in $\bar{\gamma}$ for all $(t,s) \in [0,T) \times \mathbb{R}^n_+$, then C is also a solution of Eq. (3).

This proposition is very useful for reducing the BSB equation, for which explicit solutions are very rare, to a BS equation, for which explicit solutions are common. We now show an example of this case: the Margrabe's exchange option (see [15]), whose final payoff is

$$h(s) = (s_1 - \lambda s_2)^+ \ .$$

We try to apply the results above to this option. To this end, for a generic $\gamma \in \Sigma$ we calculate the Black-Scholes price $C(t,s_1,s_2)$ of the exchange option, which is the solution of Eq. (9) when we substitute $\bar{\gamma}$ with

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \ . \tag{10}$$

By Proposition 8, if the Black-Scholes price is such that the optimization problem (7) attains its maximum at γ for all $t \in [0,T)$, $s \in \mathbb{R}^2_+$, then $C(t,s_1,s_2)$ is also a solution of the BSB equation (3).

The Black-Scholes price of this option is:

$$C(t,s_1,s_2) = s_1 N(d_1) - \lambda s_2 N(d_2) \ , \tag{11}$$

where

$$d_1 = \frac{1}{\Gamma\sqrt{T-t}} \ln\left(\frac{s_1}{\lambda s_2}\right) + \frac{1}{2}\Gamma\sqrt{T-t} \ , \quad d_2 = d_1 - \Gamma\sqrt{T-t} \ ,$$
$$\Gamma = \sqrt{(\gamma_{11}-\gamma_{21})^2 + (\gamma_{12}-\gamma_{22})^2}$$

and N is the cumulative distribution function of a centered reduced Gaussian random variable:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \, dx$$

(for a proof, see [15]). In order to apply Proposition 8, we calculate the matrix $A_{t,s}$. After some calculations (see [17]), we arrive at

$$A_{t,s} = \bar{s} \cdot D^2C(t,s) \cdot \bar{s} = \frac{s_1 N'(d_1)}{\Gamma\sqrt{T-t}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ .$$

We notice that $A_{t,s}$ is proportional $\forall (t,s) \in [0,T) \times \mathbb{R}^2_+$ to the constant matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

This means that the solution to the problem (7) is constant. Hence we easily obtain the following proposition (see [17] for the proof).

Proposition 9 *If $\bar{\gamma}$ is the maximum of $\|\gamma_1 - \gamma_2\|_{\mathbb{R}^2}$ on Σ, where γ_1, γ_2 are the rows of γ, then (11) is solution of Eq. (3).*

3 Interest rates derivatives

The general model in Section 2 finds a natural application in the superhedging of interest rates derivatives. In fact, the final payoff of interest rates derivatives can often be expressed as a deterministic function of suitable zero-coupon bonds. For example, if we define the forward LIBOR $f(t,T,\alpha)$ via the equation

$$1 + \alpha f(t,T,\alpha) = \frac{B(t,T)}{B(t,T+\alpha)}$$

the final payoff $\alpha(f(T,T,\alpha) - K)^+$ of a caplet with payment date $T+\alpha$ on the forward LIBOR $f(T,T,\alpha)$ can be written also as

$$(1+\alpha K)\left(\frac{1}{1+\alpha K} - B(T,T+\alpha)\right)^+$$

It can be proved (see for example [16]) that by changing the numeraire the caplet above is equivalent to have a payoff of

$$(1+\alpha K)\left(\frac{B(T,T)}{1+\alpha K} - B(T,T+\alpha)\right)^+$$

at time T. This means that a caplet can be seen as an exchange option between the two zero coupon bonds $B(\cdot,T)$ and $B(\cdot,T+\alpha)$. Thus a cap is simply a linear combination of exchange options between zero coupon bonds maturing at different times.

Interest rates derivatives with only one single maturity T (for example call, put and other kind of European options on zero coupon bonds, caplets and floorlets, swaps, swaptions) can be easily treated in the framework of our model; in fact, they all are characterised by a terminal payoff h and some bonds with maturity $T_1 < T_2 < \ldots < T_N$, with $T_1 = T$. In particular, the final payoff is usually writable in the form

$$h\left(\frac{B(t,T_1)}{B(t,T)}, \ldots, \frac{B(t,T_N)}{B(t,T)}\right) \tag{12}$$

In order to apply Theorem 2, we define

$$S^i_t = \frac{B(t,T_i)}{B(t,T)}$$

It follows that if the zero coupon bonds follow the dynamics

$$dB(t,T_i) = B(t,T_i)(r_t\,dt + \langle \Gamma(t,T_i), dW_t \rangle) \tag{13}$$

under $\mathbb{Q}$, then S^i follows the dynamics

$$dS_t^i = S_t^i \langle \Gamma(t, T_i) - \Gamma(t, T), dW_t' \rangle \tag{14}$$

under $\mathbb{Q}_T$, where W' is a $\mathbb{Q}_T$-Brownian motion. Then if we define $\sigma^i = \Gamma(t, T_i) - \Gamma(t, T)$ and if we assume that $\sigma = (\sigma_t^i)_t \in \mathcal{A}(\Sigma)$, then we are in the setting of section 2 and we can try to apply Theorems 3 and 6. More precisely we have the following result.

Theorem 10 *Assume that (12) and (13) hold true. Then, if the process S satisfying (14) and the payoff h defined by (12) satisfy assumptions of Theorem 3 or of Theorem 6 then $(C_t, \Delta_t)_t$ is the minimal superstrategy in the sense of Theorem 3.*

Remark 11 Observe that assumption (13) holds always true for all Gaussian models of interest rates and that in such cases Theorem 10 applies. Moreover, if we consider a non-gaussian model for interest rates we can have (13) but in such cases the diffusion coefficient is stochastic and then it is not obvious if our regularity assumptions (like compactness of the set Σ) are verified. In such cases one has to check every particular problem. Here we will consider explicitely only applications to the Gaussian case.

The case when we have more than one maturity (for example in the case of a cap or a floor) is more complex. In fact, in principle, the total payoff can be stripped in single payoffs each one having different maturities and these single payoffs can be superhedged separately with the help of the BSB equation.

Example 12 Consider a cap with dates $T_1 < \ldots < T_N$, with $T_{i+1} - T_i = \alpha$, $i = 1, \ldots, N-1$ and with single payoffs $\alpha(f(T_i, T_i, \alpha) - K)^+$ at time T_{i+1}. In [6] it was showed that in the case of a Gaussian model with stochastic volatility each caplet can be superhedged by the strategy (C^i, Δ^i) given by the BSB superstrategy, where

$$C_i(t, S_t^i, S_t^{i+1}) = S_t^i N(d_1^i) - \lambda S_t^{i+1} N(d_2^i) \tag{15}$$

where we called $\lambda = 1 + \alpha K$ and

$$d_1^i = \frac{1}{\bar{\Gamma}_i} \ln \left(\frac{S_t^i}{\lambda S_t^{i+1}} \right) + \frac{1}{2} \bar{\Gamma}_i, \quad d_2^i = d_1^i - \bar{\Gamma}_i$$

and

$$\bar{\Gamma}_i = \text{argmax}_{\;\Gamma(\cdot, T_{i+1}), \Gamma(\cdot, T_{i+1}) \in \Sigma} \int_t^{T_i} \|\Gamma(s, T_{i+1}) - \Gamma(s, T_{i+1})\| \, ds$$

This corresponds to the superhedging strategy for an exchange option.

The question is: is the superstrategy above the cheapest possible for the cap? We already know that it is the cheapest for each separate cap, but we do not know if it is the cheapest for the cap or there exists another superstrategy cheaper than this one. In the next section we will answer to this question.

4 Superreplication of caps and floors

Now we show how we can superhedge a cap with our approach. In fact our approach could be used also in a more general setup but we avoid it for brevity concentrating on this main application. We take the dates $T_1 < \ldots < T_N$, with $T_{i+1} - T_i = \alpha$, $i = 1, \ldots, N-1$. A cap is then a claim corresponding to the sum of the payoff $\alpha(f(T_i, T_i, \alpha) - K)^+$ at time T_{i+1}. In a complete market, the price of a cap at time $t < T_1$ is

$$\sum_{i=1}^{N-1} B(t, T_{i+1})(1+\alpha K)\mathbb{E}_{\mathbb{Q}_{T_{i+1}}}\left[\left(\frac{B(T_i, T_i)}{1+\alpha K} - B(T_i, T_{i+1}) \right)^+ \middle| \mathcal{F}_t \right]$$

where $\mathbb{Q}_{T_{i+1}}$ is the so-called **forward-neutral** probability for the maturity T_{i+1}. If the prices of the zero coupon bond are functions of a Markov process (for example in a short rate model like Vasicek, CIR or in a more general multifactor model), then the usual way to price a cap is this: a PDE is written to price the last caplet with maturity T_N, with terminal condition equal to the payoff of the caplet with maturity T_N; then the solution of this PDE at time T_{N-1} is summed to the payoff of the caplet with maturity T_{N-1}, and this new payoff is taken as terminal condition of a PDE pricing the last two caplets; the procedure continues in this way until time t is reached.

The procedure with our approach is similar: in fact we build a sequence of BSB equations in a similar way to obtain the price of the entire cap. More in detail, we define $s^{(i)} = (s_i, \ldots, s_N)$ and we write the i-th BSB equation as

$$\begin{cases} \dfrac{\partial C_i}{\partial t}(t, s^{(i)}) + \dfrac{1}{2}\max\limits_{\gamma\in\Sigma} \operatorname{tr}\left(D^2 C_i(t, \bar{s}^{(i)})(\bar{s}^{(i)}\gamma)(\bar{s}^{(i)}\gamma)^*\right) = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [T_{i-1}, T_i), & s^{(i)} \in \mathbb{R}_+^{N-i+1} \\ C_i(T_i, s^{(i)}) = C_{i+1}(T_i, s^{(i+1)}) + (s_i - \lambda s_{i+1})^+, & s^{(i)} \in \mathbb{R}_+^{N-i+1} \end{cases} \tag{16}$$

with $i = 1, \ldots, N-1$, where we put $C_N(T_{N-1}, s_N) \equiv 0$, so that the terminal condition of the $(N-1)$-th equation is $C_{N-1}(T_{N-1}, s_{N-1}, s_N) = (s_{N-1} - \lambda s_N)^+$.

Theorem 13 *If for all $i = 1, \ldots, N$, C_i is the solution of Equation (16) and for $t \in [T_{i-1}, T_i)$ we define $C_t = C_i(t, S_t^{(i)})$ and $\Delta_t = DC_i(t, S_t^{(i)})$, then (C, Δ) is the cheapest (in the sense of part ii) of Theorem 3) strategy that superreplicates the cap.*

Proof. We proceed by backward induction starting from $i = N$. For this case, Equation (16) reduces to Equation (3), so the thesis follows from Theorem 3. Now we assume that the thesis is true for $i+1$ and we prove that it is true for i. At time $t = T_i$, the superreplication "price" of the caplets with maturity $T_{i+1}, \ldots, T_N$ is given by $C_{i+1}(T_i, S_{T_i}^{(i+1)})$. Since the caplet with maturity T_i at time T_i has a deterministic payoff, the price of the cap at time T_i is given by $C_{i+1}(T_i, S_{T_i}^{(i+1)}) + (s_i - \lambda s_{i+1})^+$. This means that we have to superreplicate the payoff $C_{i+1}(T_i, S_{T_i}^{(i+1)}) + (s_i - \lambda s_{i+1})^+$ at time T_i. By Theorem 3, the cheapest superstrategy in $t \in [T_{i-1}, T_i)$ is given by (C, Δ) defined by $C_t = C_i(t, S_t^{(i)})$ and $\Delta_t = DC_i(t, S_t^{(i)})$, so the thesis is true also for i. The proof is complete. □

Remark 14 Observe that the result of the above Theorem 13 in fact holds every time we can apply to every single equation Theorem 10. So also for floors or other options with multiple maturity we can state a similar result. This can be done also if the set Σ is different in every equation. Moreover also the case of Σ varying mildly with time along some interval (T_{N-1}, T_N) could be treated by a straightforward extension of the results of [13, 17] (see Remark 7).

We now use Theorem 13 to answer to the question set after Example 12. By virtue of Theorem 13, in order to check if a superstrategy is the cheapest, it is sufficient to check if it is solution of the BSB equation (16). To this purpose, we rewrite Proposition 8. We define

$$F_{t,s}^{(i)}(\gamma) = \mathrm{tr}\ \left(\bar{s}^{(i)} D^2 C_i(t, s^{(i)}) \bar{s}^{(i)} \gamma\gamma^*\right) = \mathrm{tr}\ \left(A_{t,s}^{(i)} \gamma\gamma^*\right)\ , \tag{17}$$

where $A_{t,s}^{(i)} = \bar{s}^{(i)} D^2 C_i(t, s^{(i)}) \bar{s}^{(i)}$, $t \in [T_{i-1}, T_i)$ and C_i is the solution of Eq. (16).

Proposition 15 *If for all $i = 1, \dots, N$, $C_i \in C^{1,2}([0,T) \times \mathbb{R}_+^n)$ is the solution of the Black-Scholes equation*

$$\begin{cases} \dfrac{\partial C_i}{\partial t}(t, s^{(i)}) + \dfrac{1}{2}\mathrm{tr}\ (D^2 C_i(t, s^{(i)})(\bar{s}^{(i)}\bar{\gamma})(\bar{s}^{(i)}\bar{\gamma})^*) = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [T_{i-1}, T_i), \quad s^{(i)} \in \mathbb{R}_+^{N-i+1}, & \\ C_i(T_i, s^{(i)}) = C_{i+1}(T_i, s^{(i+1)}) + (s_i - \lambda s_{i+1})^+, \qquad s \in \mathbb{R}_+^{N-i+1} & \end{cases} \tag{18}$$

where $\bar{\gamma} = \mathrm{argmax}\ _{\gamma \in \Sigma} F_{t,s}^{(i)}(\gamma)$ for all $(t, s^{(i)}) \in [0,T) \times \mathbb{R}_+^{N-i+1}$, then C_i is also a solution of Eq. (16).

The proof is the same of that of Proposition 10 in [17].

Now we check if the strategy of Example 12 is the cheapest superstrategy. To this end, we calculate the quantity $A_{t,s}^{(i)}$ for the function 15. By calculations analogous to the ones of Section 6 in [17], we have that

$$A_{t,s}^i := \bar{s}^{(i)} D^2 C^i(t, s^{(i)}) \bar{s}^{(i)} = \frac{S_t^i N'(d_i)}{\Gamma_i} \begin{pmatrix} 1 & -1 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where C^i is the price of the i-th caplet given by Equation 15. By making some calculations, we have that

$$F_{t,s}^{(i)}(\gamma) = \mathrm{tr}\ \left(\sum_{i=1}^n A_{t,s}^i \gamma\gamma^*\right) = \sum_{i=1}^n \frac{S_t^i N'(d_i)}{\Gamma_i} \|\gamma_{i+1} - \gamma_i\|_{\mathbb{R}^N}^2$$

This means that if we succeed in finding $\bar{\gamma} \in \Sigma$ such that the expressions $\|\gamma_{i+1} - \gamma_i\|_{\mathbb{R}^N}$ are all maximised *simultaneously*, then the procedure in [6] works. More in detail, we have the following theorem.

Theorem 16 *If there exists $\bar{\gamma} \in \Sigma$ such that the expressions $\|\gamma_{i+1} - \gamma_i\|_{\mathbb{R}^N}$ are all maximised simultaneously, then the function $C_i(t, s^{(i)}) = \sum_{k=i}^N C^k(t, s_k, \dots, s_N)$ is solution of Equation*

(16), where C^k is given by Equation (15) and $\bar{\Gamma}_i = (T_i - t)\|\bar{\gamma}_{i+1} - \bar{\gamma}_i\|_{\mathbb{R}^N}$. Moreover, the superreplication strategy is given by

$$\begin{aligned}
C_t &= \sum_{k=i}^{N} C^k(t, S_t^1, \ldots, S_t^{N-k+2}) \\
\Delta_t^1 &= -\lambda N(d_2^N) \\
\Delta_t^k &= N(d_1^k) - \lambda N(d_2^k) \qquad \textit{for } k = 2, \ldots, N-i+1 \\
\Delta_t^{N-i+2} &= N(d_1^i)
\end{aligned}$$

Proof. The proof follows from Proposition 15 and Theorem 13. □

Remark 17 Observe that the assumption: "*there exists $\bar{\gamma} \in \Sigma$ such that the expressions $\|\gamma_{i+1} - \gamma_i\|_{\mathbb{R}^N}$ are all maximised simultaneously*" is not in general satisfied, in particular if the set Σ varies with time. However in some cases of interest like when Σ is a cube (i.e. all terms of the matrices belong to given confidence intervals) then such assumptions holds. In fact, if

$$\Sigma = \left\{ \gamma \in \mathbb{R}^{n \times n} \,:\, \gamma_{i,j} \in [\bar{\gamma}_{i,j} - \varepsilon_{i,j}, \bar{\gamma}_{i,j} + \varepsilon_{i,j}] \right\}$$

then the supremum for every $\|\gamma_{i+1} - \gamma_i\|_{\mathbb{R}^N}$ is clearly reached by the matrix

$$\gamma_0 = \begin{pmatrix} \bar{\gamma}_1 + \varepsilon_1 \\ \bar{\gamma}_2 - \varepsilon_2 \\ \bar{\gamma}_3 + \varepsilon_3 \\ \cdots \\ \bar{\gamma}_n + (-1)^{n+1}\varepsilon_n \end{pmatrix}$$

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