On the superreplication approach for European multiasset derivatives

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Abstract

In this paper we analyse the superreplication approach to stochastic volatility in the case of European multiasset derivatives. We prove that the Black-Scholes-Barenblatt (BSB) equation gives a superhedging strategy even if its solution is not twice differentiable. This is done under convexity assumptions on the final payoff h that are verified in some applications presented here. Technical issues on the existence, uniqueness and regularity of solutions of BSB equation are discussed in the Appendix.

1 Introduction

In this paper we analyse the robustness of the Black-Scholes formula for European multiasset derivatives including also cases when the value function is nonsmooth.

We give now an outline of the problem with a formulation that uses forward prices (see e.g. [8]). We have a riskless asset, whose value we suppose constant through time, and n risky assets whose prices $S = (S^1, \ldots, S^n)$ follow the dynamic

$$dS_t = \bar{S}_t \sigma_t \, dW_t,$$

where W is a n-dimensional Brownian motion under the so called forward-neutral measure $\mathbb{Q}$, the matrix process σ is adapted and takes values in a closed bounded set $\Sigma \subset M(n, n, \mathbb{R})$, and $\bar{S}_t = \text{diag}\,(S_t)$. We then consider an agent who wants to price and hedge a European contingent claim whose payoff is a deterministic function $h(\cdot)$ which is globally Lipschitz continuous, calculated in S_T. Since the market could be incomplete because of the stochastic volatility σ and the agent is not able to hedge the volatility, he chooses to hedge the option by using the superhedging approach. Following [1], [25] and [27], we assume that he fixes the price of the option as $C_t = C(t, S_t)$ and builds a self-financing portfolio consisting of a quantity $\Delta_t^i = \frac{\partial C}{\partial S_i}(t, S_t)$ of the asset $S^i, i = 1, \dots, n$, where $C(t, S)$ is a solution of a non-linear PDE, similar to the Black-Scholes equation, called Black-Scholes-Barenblatt (BSB) equation in analogy with [1] and [27]. This equation is a Hamilton-Jacobi-Bellman equation and it is linked to a stochastic control problem that have a nice financial interpretation (see section 2 and also [27]). In fact, the agent could build a "subjective" model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \, dW_t$$

where the state variable S^γ has to be controlled by the "subjective" volatility γ in order to maximize the payoff function

$$\mathbb{E}_Q \left[e^{-r(T-t)} h(S_T^\gamma) \mid \mathcal{F}_t \right]$$

corresponding to the no-arbitrage price of the contingent claim at time t. In this way, the agent wants to maximize in γ his payoff protecting himself against "the worst possible case", and the BSB equation is the Hamilton-Jacobi-Bellman equation corresponding to this optimal control problem.

The main goal of this paper is study conditions under what the above procedure can be performed, i.e. when the hedging strategy described by $\Delta_t^i = \frac{\partial C}{\partial S_i}(t, S_t)$, $i = 1, \dots, n$ is superreplicating, provided $C(t, S)$ is a solution of the BSB equation.

The main known result on this problem is substantially Theorem 3 (recalled in Section 2 for the reader convenience) that characterizes superstrategies in terms of the solutions of the BSB equation *when they are regular enough to apply Itô formula* (C^1 in time and C^2 in space). This result has been proved in the single asset case in [1] and in the multiasset case in [25] where a detailed discussion on the model and on its financial interpretation is given, (see also [27] for a simpler proof of it), and it is also shown (using known results on PDEs) that if the BSB equation is uniformly parabolic, then its solution is smooth enough to apply Itô formula. Uniform parabolicity of the BSB equation depends essentially from the set Σ being composed of invertible matrices. This is not verified in some financial examples (see Example 11 for a typical situation). For this reason it is interesting to see what can be said in non uniformly parabolic cases (see Subsections 5.3 and 5.4 for examples).

In this paper, which is an extension of a previous work [27] by one of the authors, we furnish a partial answer to the problem if there could be a Markov superhedging strategy when the BSB equation is not uniformly parabolic. In particular we show that in the general case the BSB equation admits a unique solution with first derivatives defined almost everywhere (so that the hedging strategy $\Delta_t = \frac{\partial C}{\partial S_i}(t, S_t)$ makes always sense), and we find a superhedging strategy when the payoff h is convex and the random variables S_t have a density with respect to the n-dimensional Lebesgue measure. This generalises Theorem 3 to some non-smooth cases.

The picture is still not complete, since we are not able, up to now, to show that the strategy $\Delta_t = \frac{\partial C}{\partial S_i}(t, S_t)$ is a superreplication one also in the case of non convex payoff h. This happens in some particular cases, as the example in Subsection 5.4 shows, and the general problem is at the moment open.

From the technical point of view we observe that to get existence, uniqueness and regularity results for the BSB equation we need to use the theory of viscosity solutions. The results available in the literature on this topic do not cover our case, even if the techniques can be adapted to it. We are then forced to reprove results of existence, uniqueness and regularity of viscosity solutions of the BSB equation. This a very technical part, indeed necessary, which we put in the Appendix for the reader convenience, recalling the main results in Section 3. On the other side, to prove that $\Delta_t = \frac{\partial C}{\partial S_i}(t, S_t)$ is a superreplication strategy we need to use an approximation technique with compactness arguments that holds only when h is convex. (see Section 4). We remark that we don't need (and perhaps we are not allowed because of the poor smoothness of our problem) to use stochastic integration without probability as Lyons did in [25].

The paper is organized as follows: in Section 2 we present the problem, and, for the reader's convenience we recall Theorem 3 that characterizes the superreplicating strategies in terms of a quantity R_t, that could be interpreted as a consumption rate. Moreover we prove that the strategy defined above via the BSB equation is a superreplication one. In Section 3 we present the main results on the BSB equation (proved in the Appendix); in Section 4 we prove the limit theorems about the price and the hedging portfolio, and we obtain that the portfolio defined via the BSB equation is superhedging; in Section 5 we present some examples in which the theory developed in Section 3 and 4 is used in a progressive way. The Appendix is devoted to present and prove results on existence, uniqueness and regularity for a class of Hamilton-Jacobi-Bellman equation that includes our BSB equation.

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2 The model

We suppose that there exists a riskless asset M and n risky assets S^i, $i = 1, \ldots, n$ in the market. We make the usual assumptions that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t$ represents the information available up to time t and that M and S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Besides, we assume that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is called **forward-neutral probability** [8], such that the value of the riskless asset M remains constantly equal to 1 through time, and the dynamics of the assets S^i under $\mathbb{Q}$ are the following:

$$dS_t^i = S_t^i \langle \sigma_t^i, dW_t \rangle$$

where $(W_t)_t$ is a d-dimensional $\mathbb{Q}$-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^d$ and σ^i is a d-dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\Sigma)$, where Σ is a closed bounded set in the space of $n \times d$ real matrices $M(n, d, \mathbb{R})$ and $\mathcal{A}(\Sigma)$ (which we call set of **admissible volatilities**) is the set of Σ-valued processes progressively measurable with respect to $(\mathcal{F}_t)_t$.

We can write the dynamics of the risky assets in a more compact vectorial notation in this way:

$$dS_t = \bar{S}_t \sigma_t \, dW_t$$

where:

$$\bar{S}_t = \text{diag}\,(S_t) = \begin{pmatrix} S_t^1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & S_t^n \end{pmatrix}$$

Now we consider an operator in the market that wants to create and sell a European derivative asset with payoff $h(S_T) = h(S_T^1, \ldots, S_T^n)$, where h is a locally Lipschitz continuous function having polinomial growth.

Remark 1 We do not assume that the market is complete. In particular, the filtration $(\mathcal{F}_t)_t$ could be strictly larger than the one generated by W or than the one generated by S. In this case we have a genuine stochastic volatility model and the market is incomplete. For this reason, we can assume that the number of assets n is different from the number of Brownian motions d in the model. Moreover, the assumption that the interest rate is zero can be achieved by a so-called change of numeraire, that is by expressing the prices of all the assets in the market in units of a zero coupon bond with maturity T (for more details, see [8]).

Remark 2 In applied examples, the payoff h is usually a globally Lipschitz continuous function so its growth is at most linear. We decided to work with the more general case of polinomially growing h for expository reasons; in fact most of our results holds for h polinomially growing (when the global Lipschitz property of h will be needed we will put it as an extra assumption).

In order to price and hedge the asset, the agent fixes a price C_t for the asset at time t and builds a self-financing portfolio Π, holding η_t units of the money market account M and Δ_t^i units of the asset S^i at time t. We indicate with Π_t the value of the portfolio at time t, defined by:

$$\Pi_t = \eta_t + \langle \Delta_t, S_t \rangle \tag{1}$$

where $\Delta_t = (\Delta_t^1, \dots, \Delta_t^n)$. In order to prevent arbitrage opportunities, we say that the portfolio is **admissible** if Π is a supermartingale: in this way, if $\Pi_0 = 0$, we have $\mathbb{E}[\Pi_T] \leq 0$, so there aren't arbitrage opportunities in the market. We say that the portfolio is **self-financing** if Π follows the dynamic

$$\begin{cases} d\Pi_t = \langle \Delta_t, dS_t \rangle \\ \Pi_0 = C_0 \end{cases} \tag{2}$$

If the portfolio is self-financing, the process Δ is sufficient to characterize it, and $\eta_t = \Pi_t - \langle \Delta_t, S_t \rangle$.

The market is not complete because of the stochastic volatility σ, so the agent chooses to hedge the claim using the superhedging approach. To this aim, we define the **tracking error**:

$$e_t = \Pi_t - C_t$$

so $e_0 = 0$ by definition of Π. Intuitively, the tracking error gives the error made by the operator in estimation, or better, the difference between the hedging portfolio held by the agent and the option sold. If the portfolio is admissible and self-financing, we say that (C, Δ) is a

- **superhedging strategy**, or simply **superstrategy** if e is a non decreasing process
- **subhedging strategy**, or simply **substrategy**, if e is a non increasing process
- **hedging strategy** if e is identically equal to zero.

The idea behind these definitions is the following: if one builds a superstrategy, one can successfully hedge a short position in the contingent claim C_t, while if one builds a substrategy, one can successfully hedge a long position in C_t. In particular with a superstrategy we have that $e_T \geq 0$, that is $\Pi_T \geq C_T = h(S_T)$, so the portfolio succeeds in overhedging the contingent claim. We notice that superstrategies and substrategies are good candidates to be arbitrages: in fact if we have (say) $\Pi_T \geq C_T$ $\mathbb{Q}$-a.s. and $\Pi_T > C_T$ with positive probability, then the agent succeeds in making a profit with no initial endowment. This means that the value C_t in a superstrategy has to be interpreted as an arbitrage upper bound for the price of the claim at time t; similarly, the value C_t in a substrategy has to be interpreted as a lower bound. In fact, if the price of the claim is C_t or less, one could build an arbitrage by buying the claim and by selling short the subhedging portfolio for the same price, thus making the final profit $C_T - \Pi_T = -e_T \geq 0$.

As already outlined by [1], [9], [25] and [27], in order to have a superstrategy, a natural procedure is this: the agent fixes the price C_t of the option and the quantities Δ_t of the risky assets in the hedging portfolio Π at time t as

$$C_t = C(t, S_t), \qquad \Delta_t^i = \frac{\partial C}{\partial s_i}(t, S_t)$$

where C is the solution of the following partial differential equation:

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \operatorname{tr}\left(D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*\right) = 0, & t \in [0,T), s \in \mathbb{R}_+^n \\ C(T,s) = h(s), & s \in \mathbb{R}_+^n \end{cases} \tag{3}$$

where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s), \ldots, \frac{\partial C}{\partial s_n}(t,s))$, and $D^2C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (3), as in [1], will be called from now on the **Black-Scholes-Barenblatt (BSB) equation** for h. Moreover, in order to have a substrategy it is sufficient to proceed as above by substituting the maximum operator in Equation (3) with a minimum.

This choice has a stochastic control interpretation: we assume that the agent doesn't know the volatility σ, but in order to hedge the asset he can use the model

$$dS_\tau^\gamma = \bar{S}_\tau^\gamma \gamma_\tau \, dW_\tau; \qquad S_t^\gamma = S_t^\sigma. \tag{4}$$

Here the initial datum S_t^σ is the present market price of the assets at time t which depends of course on the real but unknown volatility σ up to time t. The process $\gamma \in \mathcal{A}_t(\Sigma)$ is another admissible volatility starting from time t, where we set $\mathcal{A}_t(\Sigma)$ the set of admissible volatilities after time t, i.e. the set of processes : $[t,T] \mapsto \Sigma$ progressively measurable with respect to $(\mathcal{F}_u)_{u \geq t}$. We can interpret the γ as a control, corresponding to the subjective volatility of the agent, that he uses in order to decide his strategy. If the market were complete and the dynamics were given by Equation (4), then the arbitrage free price of the European contingent claim would be

$$C_t^{\mathbb{Q},\gamma} = \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \;\middle|\; S_t^\sigma\right] \tag{5}$$

where $\mathbb{Q}$ would be the unique forward-neutral measure of the problem. Since the market is not complete, we are not sure that the contingent claim defined by h is attainable. In particular, we have two sources of uncertainty, that are the particular forward-neutral measure $\mathbb{Q}$ and the volatility γ. Since the agent doesn't know either of these objects, he associates the price of the claim with the interval

$$\left\{ \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma)\middle| \mathcal{F}_t\right] \;\middle|\; \mathbb{Q} \simeq \mathbb{P} \text{ such that Eq. (4) holds under } \mathbb{Q} \text{ and } \gamma \in \mathcal{A}(\Sigma) \right\}$$

in the real line, called the **set of admissible prices**. However, since $\gamma \in \mathcal{A}(\Sigma)$, for all the forward-neutral measures $\mathbb{Q}$ and $\gamma \in \mathcal{A}(\Sigma)$ we have, setting $C_t^{\mathbb{Q},\gamma} = \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \;\middle|\; S_t^\sigma\right]$,

$$V_{\mathbb{Q}}^-(t, S_t^\sigma) \leq C_t^{\mathbb{Q},\gamma} \leq V_{\mathbb{Q}}^+(t, S_t^\sigma)$$

where

$$V_{\mathbb{Q}}^-(t,s) = \inf_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \middle| S_t^\gamma = s\right] \tag{6}$$

and

$$V_{\mathbb{Q}}^+(t,s) = \sup_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \middle| S_t^\gamma = s\right] \tag{7}$$

are the value functions of two optimal control problems on the reference probability system $\nu = (\Omega, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{Q}, W)$ (see Appendix A.2 for details), having as payoff function the right-hand side of Equation (5). In particular, under the assumptions we made, $V_{\mathbb{Q}}^+$ is the unique viscosity solution of Equation (3) (see Theorem 6 in Section 3) and does not depend on the particular choice of the reference probability system ν, so we can drop the subscript in $\mathbb{Q}$. Thus we have that $[V^-(t, S_t^\sigma), V^+(t, S_t^\sigma)]$ defines the interval of admissible arbitrage prices for the claim at time t, in the sense that if the price of the claim lies outside this interval, then it is possible to build an arbitrage in the market. This can be interpreted as follows: when the agent decides the price of the claim, he wishes to protect himself against the worst possible outcome, and he chooses the price that corresponds to this. Moreover, we can interpret the γ as a control, corresponding to the subjective volatility of the agent, that he uses in order to decide his strategy.

We notice that the substrategy case is in some sense symmetric to the superstrategy one. This has also a simple mathematical explanation: in fact, if C is the solution of the BSB equation with final condition h, then $-C$ is the solution of the BSB equation with min instead of max, and final condition $-h$. In other words, if C defines a superstrategy for the claim defined by h, then $-C$ defines a substrategy for the claim defined by $-h$. This allows us to analyse only one of the two cases, and to have automatically results for the other. For this reason, in the rest of the paper we will analyse only the superstrategy case, and it will be implicit that analogous results will hold also for substrategies.

We now present the main known result on this subject. We indicate with $C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$ the space of functions C that are continuous and with polynomial growth on $[0,T) \times \mathbb{R}_+^n$ together with their first derivative in t and first and second derivatives in s. The Theorem 3 in the form below is stated and proved in [27] (see also [1], [25] for previous similar results).

Theorem 3 *If we restrict ourselves to the case $C_t = C(t, S_t)$, with $C \in C_p^{1,2}\left([0,T) \times \mathbb{R}_+^n\right)$, then:*

i) (C, Δ) is a superstrategy if and only if $\Delta_t = DC(t, S_t)$ and

$$R_t = -\frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2}\mathrm{tr}\; D^2C(t, S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*$$

is non negative $\mathbb{Q}$-a.s. for all $t \in [0,T]$.

ii) If C is the solution of Equation (3) and $\Delta_t = DC(t, S_t)$, then (C, Δ) is a superstrategy. Moreover there is not any superstrategy (C', Δ') such that $C_t' < C(t, S_t^\gamma)$ with positive probability for some t and for all $\gamma \in \mathcal{A}(\Sigma)$.

This is a characterization theorem of the superstrategies. In fact, provided the value function $C \in C_p^{1,2}$, the theorem gives the minimal superstrategy, in the sense that there doesn't exist a cheaper strategy that succeeds in giving an increasing tracking error for all $\gamma \in \mathcal{A}(\Sigma)$. Since the agent doesn't know the true volatility σ, he must protect himself against all the possible volatilities $\gamma \in \mathcal{A}(\Sigma)$ and this is the best result he can expect.

Remark 4 The quantity R_t can be seen as a consumption instantaneous rate obtained by the superstrategy of the agent. A discussion on the financial interpretation of R can be found in [25].

For completeness we give also the results for a substrategy.

Theorem 5 *If we restrict ourselves to the case $C_t = C(t, S_t)$, with $C \in C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$, then:*

i) *(C, Δ) is a substrategy if and only if $\Delta_t = DC(t, S_t)$ and*

$$R_t = -\frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2}\mathrm{tr}\ D^2C(t, S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*$$

is non positive $\mathbb{Q}$-a.s. for all $t \in [0,T]$.

ii) *If C is the solution of Equation (3) (provided we substitute the* max *with* min*), and $\Delta_t = DC(t, S_t)$, then (C, Δ) is a substrategy. Moreover there does not exist another superstrategy (C', Δ') such that $C'_t < C(t, S_t^\gamma)$ with positive probability for some t and for all $\gamma \in \mathcal{A}(\Sigma)$.*

We have seen above that in order to have a Markov superstrategy, we need a $C^{1,2}$ solution of Equation (3). This requirement however is very strong, see e.g. [14], p.162. In the following section we will see what kind regularity we can expect for the solution of Equation (3), while in Section 4 we see how to prove a weak version of Theorem 3 in the case when we do not know if the solution of (3) is $C^{1,2}$.

3 The BSB equation

We collect here main results on existence, uniqueness and regularity that are presented and proved in the Appendix. We point out that most of them are known, or at least expected, in the literature on Partial Differential Equations (PDE's). However we were not able to find precise reference for many of them. For this reason we decided to devote the Appendix of the paper to state and prove precise results for a family of PDE's that includes Equation (3). Here we recall only what is needed to study our financial problem.

The first step is to recall a general result of existence and uniqueness of viscosity solutions for the BSB equation (3) (see Theorem 23 and Corollary 30) that always holds true in our case.

Theorem 6 *Let Σ be compact, h be locally Lipschitz continuous and h, Dh have polynomial growth. Then the value function $C = V^+$ defined by Equation (7) is a viscosity solution of Equation (3) in $[0,T] \times \mathbb{R}_+^n$. Moreover, C is the unique viscosity solution having polynomial growth that satisfies the boundary condition $C(T, s) = h(s)$ for all $s \in \mathbb{R}_+^n$.*

The second step is to prove smoothness of the viscosity solution. We have first the following general result on the existence of the first weak derivative (see Theorem 31 in the Appendix).

Theorem 7 *Let Σ be compact, h be locally Lipschitz continuous and h, Dh have polynomial growth. Then, for every $p \in [1, +\infty]$ we have $D_s V \in L^p_{\text{loc}}([0,T] \times \mathbb{R}^n_+)$ and for suitable $M > 0$*

$$|D_s V(t,s)| \leq M(1 + |s|^k)$$

for almost every $(t,s) \in [0,T] \times \mathbb{R}^n_+$.

Now we go to stronger regularity results involving the second derivative that comes from known results on uniformly parabolic equations. The main references on regularity for uniformly parabolic equations are [7] and [30]. In order to use these results, we have to make a change of variable to transform the BSB equation (which is never uniformly parabolic) into another one. We put $y_i = \log s_i$. Then the BSB equation becomes:

$$\begin{cases} \dfrac{\partial \tilde{C}}{\partial t}(t,y) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \operatorname{tr}((D^2 \tilde{C}(t,y) - \operatorname{diag}(D\tilde{C}(t,y)))\gamma\gamma^*) = 0 \\ \\ \tilde{C}(T, \cdot) = \tilde{h}(\cdot) \end{cases} \tag{8}$$

where $\tilde{C}(t,y) = C(t,(e^{y_i})_i) = C(t,s)$. In fact we can see that C is a viscosity solution of Eq. (3) if and only if $\tilde{C}$ is a viscosity solution of Eq. (8) (see [2, p. 38-39] for change of variables in the theory of viscosity solutions). We notice that in the BSB equation (3) the domain is $s \geq 0$, while in this equation the domain is all $\mathbb{R}^n$. In order to use results of [7] and [30], we have to recall the following.

Definition 8 We say that Equation (8) is **uniformly parabolic** if there exists $M > m > 0$ such that for all $\gamma \in \Sigma$ and $\xi \in \mathbb{R}^n$,

$$m\|\xi\|^2 \leq \langle \gamma\gamma^* \xi, \xi \rangle \rangle \leq M\|\xi\|^2$$

The above definition can be rewritten in this way:

$$\exists M' > m' > 0 \text{ such that } \forall \gamma \in \Sigma, \xi \in \mathbb{R}^n, m'\|\xi\| \leq \|\gamma^* \xi\| \leq M'\|\xi\|$$

This implies that if there exists a $\gamma \in \Sigma$ which has not rank equal to n, then Equation (8) is not uniformly parabolic. In fact if we take $\xi \in \ker \gamma^*$, then $\|\gamma^* \xi\| = 0$. We show that also the converse is true. To this purpose, we use the notation $\Sigma^2 = \{\gamma\gamma^* | \gamma \in \Sigma\}$.

Lemma 9 *If Σ is closed and bounded in $M(n,n,\mathbb{R})$, then Eq. (8) is uniformly parabolic if and only if $\Sigma^2 \subseteq \mathrm{GL}(n,\mathbb{R})$, where*

$$\mathrm{GL}(n,\mathbb{R}) = \{\gamma \in \mathbb{R}^{n \times n} \mid \det \gamma \neq 0\}$$

Proof. As seen before, if $\exists \gamma \in \Sigma$ such that $\gamma\gamma^* \notin \mathrm{GL}(n,\mathbb{R})$, then Eq. (8) is not uniformly parabolic. Conversely, if $\Sigma^2 \subseteq GL(n,\mathbb{R})$, then $\gamma\gamma^*$ is positive definite $\forall \gamma \in \Sigma$; besides, the function that goes from γ to the least eigenvalue of $\gamma\gamma^*$ is continuous, so it has a minimum $c' > 0$; also the function going from γ to the greater eigenvalue of $\gamma\gamma^*$ is continuous, so it has a maximum $C' > 0$. Since Σ is a compact set in $M(n,d,\mathbb{R})$, the result follows. $\square$

We notice that if $d < n$, then Eq. (8) can not be uniformly parabolic, because $\gamma\gamma^*$ has at most rank d for $\gamma \in \Sigma$.

The above Lemma 9 allows us to use results of [7] and [30] to state the following theorem.

Theorem 10 *Let Σ be compact, h be locally Lipschitz continuous and h, Dh have polynomial growth. If $\Sigma^2 \subseteq \mathrm{GL}(n, \mathbb{R})$ then there exists $\alpha \in (0,1]$ such that the viscosity solution C of the BSB equation (3) belongs to $C^{2,\alpha}(\mathbb{R}^n)$.*

Proof. We do not give a real proof, but we just recall what kind of results can be used to get our claim giving references of them. Thanks to Lemma 9, Equation (8) is uniformly parabolic and it has a unique viscosity solution, which we denote by $\tilde{C}$. Since all the coefficients are uniformly Hölder continuous (i.e. uniformly bounded in C^β for some β), one can prove that the solution is in $C^{2,\alpha}$ for some α (see [7] and the discussion before Theorem 9.1). This follows substantially in two steps. The first one is to apply a result of [7] that states $C^{1,\alpha}$ regularity for the solutions of Equation (8) when all the coefficients are uniformly Hölder continuous. The second one is to use a theorem of Wang [30] that gives $C^{2,\alpha}$ regularity for equations of the form $-u_t + F(D_s^2 u) = f$, with f belonging to $C^{1,\alpha}$ and F convex. This theorem can be used thanks to the $C^{1,\alpha}$ regularity results of [7] since then the gradient is just a Hölder continuous function that can be incorporated into the right hand side f. Then there exists $\alpha \in (0,1]$ such that $\tilde{C} \in C^{2,\alpha}(\mathbb{R}^n)$.[7]. This means that also $C(t, s_1, \ldots, s_n) = \tilde{C}(t, e^{y_1}, \ldots, e^{y_n})$ is $C^{2,\alpha}$, so we get the claim. □

As recalled in Section 2 the above regularity results allows us to apply the Itô formula and so Theorem 3. The final conclusion is that if the set Σ^2 is a closed bounded set in $\mathrm{GL}(n, \mathbb{R})$, then there exists a Markov superstrategy, and it is given by the space derivative of the solution of Equation (3). Conversely, if this not happen, then we cannot apply the Theorem 10 above, so we cannot say that $C \in C^{1,2}$ (nor apply Theorem 3) without proving new regularity results that are unknown at this stage. However this case is interesting from the financial point of view as the following example, taken from [27], shows.

Example 11 Consider the case of 2 assets, each one with known volatility but with mutual correlation unknown, which we assume to follow a stochastic process. Then

$$\Sigma = \left\{ \gamma \in M(n,n,\mathbb{R}) \;\middle|\; \gamma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}, \quad \rho \in [-1,1] \right\}$$

As indicated in [27], this corresponds to the case when the volatilities σ_1, σ_2 of the two assets S_1 and S_2 are known, but the correlation is not. In particular we allow S_1 and S_2 to be in all the states between perfectly positively correlated ($\rho = 1$) and perfectly negatively correlated ($\rho = -1$), passing through a null correlation ($\rho = 0$). We can see that for $\rho = \pm 1$ we have that $\gamma\gamma^* \notin \mathrm{GL}(n, \mathbb{R})$ and the BSB equation (3) is not uniformly parabolic. This is a typical situation arising when we want to price options on two correlated assets. In this case typically one has a good estimate for σ_1 and σ_2 but not for the correlation ρ which is more difficult to estimate.

Remark 12 The BSB equation (3) can be also formulated by adding suitable boundary conditions when $s = 0$ and $s \to +\infty$ (see on this [31, p. 47]). We did not put these conditions here since they are automatically satisfied in our case by the viscosity solutions (see e.g. [3, 14, 15])

4 The superstrategy

In this section we show how to prove a weaker version of Theorem 3 in the case when uniform parabolicity does not hold. For sake of simplicity, we make the assumption that

$d = n$, but the result remains true (with heavier notation) also for $d > n$. In particular we will prove that also in this case the process (C, Δ) defined of Section 2 gives a superstrategy when the final payoff h is a convex (or semiconvex) function of the assets and the law of the price process S_t is absolutely continuous with respect to the Lebesgue measure. This case doesn't cover all the possible payoffs (for example, it doesn't cover call-spreads), but indeed many of the multiasset European options (call options on the maximum, call and put options on the mean, exchange options, etc.) are covered.

We observe that the analogous results for a substrategy needs to assume that h is concave, not convex and so the symmetry between sub and super strategies results here is broken. In fact, most of the examples require h to be convex (and generally not concave). So, if we want to get a result on existence of substrategies when h is convex we need the additional assumption that the set Σ be convex (see Remark 16 below).

In order to get the result, we approximate the solution C of the non-uniformly parabolic BSB equation (3) with a sequence $(C_{\varepsilon_n})_n$, where C_ε is the solution of a BSB equation having $\Sigma_\varepsilon = \Sigma + \varepsilon I$ and such that the equation is uniformly parabolic. Then the functions C_ε converges pointwise to C, because these functions are all value functions of suitable optimal control problem. Besides, we prove an estimate on the weak first derivatives of C_ε and C, based on estimates on the function h. This, joint to the fact that the C_ε and C are convex (thanks to the particular structure of the control problem), gives us a compactness result in $W^{1,p}_{\text{loc}}$, that allows us to obtain a sequence $(C_{\varepsilon_n})_n$ such that $D_s C_{\varepsilon_n}$ converges in L^p to $D_s C$. This convergence result implies that also the hedging portfolio Π_t is the pointwise limit of portfolios Π^n_t defined via the functions C_{ε_n}. This convergence implies the convergence of the tracking errors e^n_t of the approximating portfolios to the tracking error e_t; since the e^n are increasing processes, this means that e is so for all $t \in (0, T]$, so we have a superhedging strategy also in this case.

We start by recalling our assumptions

Hypothesis 13 *(i) The payoff h is convex and Lipschitz continuous ($\|h\|_{\text{Lip}}$ will denote its Lipschitz constant).*

(ii) For every $t > 0$ the law of the random variable S^σ_t has a density with respect to the Lebesgue measure which is absolutely continuous.

Remark 14 We recall that all the results below hold true (with more or less standard arguments) if the function h is only semiconvex and polinomially growing, locally Lipschitz. However, since this extension has not a big interest in financial examples and would make more complicated the readability of the proofs, we do not treat it.

For each $\varepsilon \in (0, 1)$, we consider the equation

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\max_{\gamma \in \Sigma_\varepsilon} \operatorname{tr}(D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*) = 0 \\ \\ C(T,\cdot) = h(\cdot) \end{cases} \tag{9}$$

where $\Sigma_\varepsilon = \Sigma + \varepsilon I$. This corresponds to the control problem having as controlled process

$$dS^{\gamma,\varepsilon}_t = S^{\gamma,\varepsilon}_t \gamma_t \, dW_t + S^{\gamma,\varepsilon}_t (\varepsilon I)\, d\overline{W}_t \tag{10}$$

with $\gamma \in \mathcal{A}(\Sigma)$, and as value function

$$C_\varepsilon(t,s) = \sup_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}\left[\, h(S_T^{\gamma,\varepsilon}) \middle|\, S_t^{\gamma,\varepsilon} = s\right] \tag{11}$$

where $\overline{W}_t$ is an n-dimensional Brownian motion independent of $\mathcal{F}_t$. To be precise, here we consider a reference probability system $\mu = (\Omega \times \Omega', (\mathcal{F}'_s)_{s\in[t,T]}, \mathbb{Q} \times \mathbb{Q}', (W, \bar{W}))$, where Ω' is the canonical Wiener space of $\bar{W}$, $\mathcal{F}'_s = \mathcal{F}_s \vee \sigma(\bar{W}_u, u \le s)$ and $\mathbb{Q}'$ is the Wiener measure of $\bar{W}$. However, to avoid this heavy notation, we will indicate this enlarged reference probability system as usual by $\nu = (\Omega, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{Q}, (W, \bar{W}))$. We have that $S_t^{\gamma,\varepsilon} = S_t^\gamma \mathcal{E}_t(\varepsilon\overline{W})$, where $\mathcal{E}_t(\varepsilon\overline{W}) = \exp(\varepsilon\overline{W}_t - 1/2\varepsilon^2 t)$ is the Doleans-Dale exponential of the process $\varepsilon\overline{W}$ calculated in t.

We now show the regularity of the solutions of equation (9).

Lemma 15 *The function C_ε is the unique viscosity solution of Equation (3) on the class $C_p([0,T] \times \mathbb{R}^n_+)$. Moreover it is convex and of class $C^{2,\alpha}$ with respect to s.*

Proof. Existence, uniqueness and regularity follow from Theorems 6 and 10 of Section 3.

To prove convexity we use substantially that the supremum of convex function is convex and the linearity of the equation for the price process $S_t^{\gamma,\varepsilon}$. In fact, denoting by $S_u^{\gamma,\varepsilon}(t,s)$ the solution of Equation (10) with initial condition $S_t^{\gamma,\varepsilon} = s$, we have that for $\lambda \in [0,1]$ and $s_1, s_2 \in \mathbb{R}^n$,

$$S_u^{\gamma,\varepsilon}(t, \lambda s_1 + (1-\lambda)s_2) = \lambda S_u^{\gamma,\varepsilon}(t,s_1) + (1-\lambda) S_u^{\gamma,\varepsilon}(t,s_2)$$

Then by definition of C_ε and by using the linearity of the equation for the price process $S_t^{\gamma,\varepsilon}$,

$$\begin{aligned}
C_\varepsilon(t, \lambda s_1 + (1-\lambda)s_2) &= \sup_{\gamma\in\Sigma} \mathbb{E}\left[h(S_T^{\gamma,\varepsilon}(t, \lambda s_1 + (1-\lambda)\, s_2)\right] = \\
&= \sup_{\gamma\in\Sigma} \mathbb{E}\left[h\left(\lambda S_T^{\gamma,\varepsilon}(t,s_1) + (1-\lambda) S_T^{\gamma,\varepsilon}(t,s_2)\right)\right] \le \\
&\le \sup_{\gamma\in\Sigma} \mathbb{E}\left[\lambda h\left(S_T^{\gamma,\varepsilon}(t,s_1)\right) + (1-\lambda) h\left(S_T^{\gamma,\varepsilon}(t,s_2)\right)\right] \le \\
&\le \lambda \sup_{\gamma\in\Sigma} \mathbb{E}\left[h\left(S_T^{\gamma,\varepsilon}(t,s_1)\right)\right] + (1-\lambda) \sup_{\gamma\in\Sigma} \mathbb{E}\left[h\left(S_T^{\gamma,\varepsilon}(t,s_2)\right)\right]
\end{aligned}$$

which gives the claim. □

Remark 16 We observe that in the case of substrategy, to get convexity of C_ε we would need a further assumption: the set Σ be convex. In fact in this case the last step would not work since we have inf instead of sup. Convexity of Σ ensures that, for every given admissible strategy γ_1 and γ_2 we can find an admissible γ_λ such that

$$S_T^{\gamma_\lambda,\varepsilon}(t, \lambda s_1 + (1-\lambda)s_2) = \lambda S_T^{\gamma_1,\varepsilon}(t,s_1) + (1-\lambda) S_T^{\gamma_2,\varepsilon}(t,s_2).$$

This allows to repeat the final step of the prove above. Then, under the assumption of convexity of Σ, all the results of this section can be repeated for the substrategy case.

Proposition 17 *If we call C_ε the viscosity solution of Eq. (9) and C the viscosity solution of Eq. (3), then $C_\varepsilon \to C$ pointwise for $\varepsilon \to 0$, and there exists a subsequence $(\varepsilon_k)_k$ such that $D_s C_{\varepsilon_k}$ converges to $D_s C$ in $L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^n_+)$ and Lebesgue-almost surely in $(0,T) \times \mathbb{R}^n_+$.*

In the proof, we will use a lemma, due to Lions, which we present in a slightly different formulation:

Lemma 18 *Let $(u_n)_{n>0}$ be a bounded family of concave functions in $L^\infty([0,T]\times B)$, with B a bounded set in $\mathbb{R}^n_+$, such that $(D_s u_n)_n$ is bounded in $L^\infty([0,T]\times B;\mathbb{R}^n)$. Then $(D_s u_n)_n$ is relatively compact in $L^p([0,T]\times B)$ for all $p\in[1,\infty)$.*

Proof. The proof is the same as in [22], pag. 209, by noticing that in that proof the time derivatives $D_t u_n$ has no role. □

Now we prove our proposition.

Proof.

- $C_\varepsilon \to C$ pointwise. We show that the difference $C_\varepsilon(t,s)-C(t,s)$ goes to 0 as $\varepsilon\to 0$. We will show only one inequality as the other one can be obtained exactly with the same procedure. Taking γ_δ a δ-suboptimal strategy for $C_\varepsilon(t,s)$ we have

$$\begin{aligned} C_\varepsilon(t,s)-C(t,s) &= \sup_{\gamma\in\Sigma}\mathbb{E}\left[h(S_T^{\gamma,\varepsilon}(t,s)\right]-\sup_{\gamma\in\Sigma}\mathbb{E}\left[h(S_T^{\gamma}(t,s)\right]\leq \\ &\leq \delta+\mathbb{E}\left[h(S_T^{\gamma_\delta,\varepsilon}(t,s)\right]-\mathbb{E}\left[h(S_T^{\gamma_\delta}(t,s)\right]\leq \\ &\leq \delta+\mathbb{E}\left[\|h\|_{\mathrm{Lip}}\left|S_T^{\gamma_\delta}(t,s)(\mathcal{E}(\varepsilon\overline{W})-1)\right|\right]\leq \\ &\leq \delta+\|h\|_{\mathrm{Lip}}\mathbb{E}\left[\left|S_T^{\gamma_\delta}(t,s)\right|^2\right]^{1/2}\cdot\mathbb{E}\left[\left|\mathcal{E}(\varepsilon\overline{W})-1\right|^2\right]^{1/2} \end{aligned}$$

  Since we have the estimate (see Appendix)

$$\mathbb{E}\left[|S_T^{\gamma_\delta}(t,s)|^2\right]\leq B_2(1+|s|^2)$$

  which is independent of δ and ε, the claim follows by the mean square convergence to 0 of the random variable $\mathcal{E}(\varepsilon\overline{W})-1$ as ε tends to 0. Note that the convergence that we obtain is uniform on bounded sets of $[0,T]\times\mathbb{R}^n_+$.

- $D_sC_\varepsilon \xrightarrow{L^p_{\mathrm{loc}}} D_sC$: we have that $\|D_sC_\varepsilon(t,s)\|_\infty\leq\|h\|_{\mathrm{Lip}}$ uniformly in ε. If we fix a compact set B in $\mathbb{R}^n_+$ and $p\in[1,+\infty)$, then the sequence $(D_sC_{1/n})_n$ is bounded in $L^\infty([0,T]\times B)$, and the functions $-C_{1/n}$ are concave. Then, by Lemma (18), $(D_sC_{1/n})_n$ is relatively compact in $L^p([0,T]\times B)$, so there is a subsequence $(D_sC_{\varepsilon_k})_k$ and a $f\in L^p(B)$ such that $-D_sC_{\varepsilon_k}\to f$ in $L^p(B)$. Besides, for all $g\in C_0^\infty(B)$ we have that

$$\langle D_sC_{\varepsilon_k},g\rangle_{L^p(B)}=-\langle C_{\varepsilon_k},D_sg\rangle_{L^p(B)}\to-\langle C,D_sg\rangle_{L^p(B)}=\langle D_sC,g\rangle_{L^p(B)}$$

  Since $C_0^\infty(B)$ is dense in $L^p(B)$, we obtain that $D_sC_{\varepsilon_k}\xrightarrow{L^p_{\mathrm{loc}}}D_sC$. This implies that there exists another subsequence, which we still call $(\varepsilon_k)_k$ such that $D_sC_{\varepsilon_k}\longrightarrow D_sC$ a.s. in $(0,T)\times\mathbb{R}^n$. This completes the proof.

□

Now we analyse the tracking error process $e_t=\Pi_t-C_t$. We call for brevity $C_k=C_{\varepsilon_k}$ and recalling that the true (but unknown) volatility is denoted by σ we will write S_t^k in place of $S_t^{\sigma,\varepsilon_k}$. Now, calling $C_t^k=C_{\varepsilon_k}(t,S_t^k)$ and Π^k the corresponding superreplicating portfolio, the quantity $e_t^k=\Pi_t^k-C_t^k$ is the tracking error for the approximating problem.

By Lemma 15 and Theorem 3, the e^k are increasing processes (note that this problem is completely artificial here and is used only to get properties in the limit). To conclude we need to show that e_t^k converges a.s. to e_t for $k \to \infty$, (since the limit a.s. of nondecreasing processes is still non decreasing a.s.).

We now prove the convergence results.

Theorem 19 *If Hypothesis 13 holds then $C^k \to C$ and $\Pi^k \to \Pi$ a.s., uniformly in $t \in [0,T]$ on a suitable subsequence.*

Proof.

Part 1. $C_t^k \to C_t$ a.s..

Recall that, thanks to Proposition 17 the functions $(C_k)_k$ converges to C uniformly on bounded subsets of $[0,T] \times \mathbb{R}_+^n$. Then, for every $R > 0$ there exists a modulus ω_R such that

$$\sup_{t\in[0,T],|s|\le R} |C_k(t,s) - C(t,s)| \le \omega_R(1/k)$$

so that, for suitable $M_1 > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |C_k(t,S_t^k) - C(t,S_t^k)|\right] \le \\
&\le \omega_R(1/k) + \mathbb{E}\left[\sup_{t\in[0,T]} |C_k(t,S_t^k) - C(t,S_t^k)| \mathbf{1}_{S_t^k > R}\right] \le \\
&\le \omega_R(1/k) + \mathbb{E}\left[\sup_{t\in[0,T]} |C_k(t,S_t^k) - C(t,S_t^k)| \mathbf{1}_{\sup_{t\in[0,T]} S_t^k > R}\right] \le \\
&\le \omega_R(1/k) + \mathbb{E}\left[\sup_{t\in[0,T]} |C_k(t,S_t^k) - C(t,S_t^k)|^2\right]^{1/2} \mathbb{Q}\{\|S_\cdot^k\| > R\}^{1/2} \le \\
&\le \omega_R(1/k) + \frac{M_1}{R}\mathbb{E}[(1+\|S_\cdot^k\|)^2]^{1/2}\mathbb{E}[\|S_\cdot^k\|^2]^{1/2}
\end{aligned}$$

Using the estimate above we get that, for every $R > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |C_t^k - C_t|\right] \le \\
&\le \mathbb{E}\left[\sup_{t\in[0,T]} |C_k(t,S_t^k) - C(t,S_t^k)|\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |C(t,S_t^k) - C(t,S_t^\sigma)|\right] \\
&\le \omega_R(1/k) + M_2\frac{M_1}{R}\mathbb{E}[(1+\|S_\cdot^k\|)^2]^{1/2}\mathbb{E}[\|S_\cdot^k\|^2]^{1/2} + M_3\mathbb{E}[\|S_\cdot^\sigma(\mathcal{E}_\cdot(\varepsilon_k\overline{W}) - 1)\|]
\end{aligned}$$

for suitable $M_2, M_3 > 0$. Taking R sufficiently large and then letting $k \to \infty$ we get

$$\lim_{k\to+\infty} \mathbb{E}\|C_t^k - C_t\| = 0.$$

This gives the a.s. uniform convergence on a suitable subsequence.

Part 2. $\Pi_t^k \to \Pi_t$.

We have

$$\Pi_t^k = \int_0^t \langle D_s C_k(u,S_u^k), \bar{S}_u^k(\sigma_u\, dW_u + \varepsilon_k \overline{W}_u)\rangle + C_0$$

and

$$\Pi_t = \int_0^t \langle D_s C(u, S_u^\sigma), \bar{S}_u^\sigma \sigma_u \; dW_u \rangle + C_0$$

so

$$\begin{aligned}
\Pi_t^k - \Pi_t &= \int_0^t \langle D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma), \bar{S}_u^k \sigma_u \; dW_u \rangle + \\
&+ \int_0^t \langle D_s C_k(u, S_u^\sigma) - D_s C(u, S_u^\sigma), \bar{S}_u^k \sigma_u \; dW_u \rangle + \\
&+ \int_0^t \langle D_s C(u, S_u^\sigma), \left[\bar{S}_u^k - \bar{S}_u^\sigma \right] \sigma_u \; dW_u \rangle + \int_0^t \langle D_s C_k(u, S_u^k), \bar{S}_u^k \varepsilon_k I \; d\overline{W}_u \rangle
\end{aligned}$$

We now denote with (i), (ii), (iii) and (iv) the terms on the right hand side, and estimate their mean square. For the first one, setting $\|\Sigma\| = \sup_{\sigma \in \Sigma} \|\sigma\|_{M(n,n;\mathbb{R})}$ we have:

$$\begin{aligned}
\mathbb{E}[|(i)|^2] &\le \|\Sigma\|^2 \mathbb{E} \int_0^t \|\bar{S}_u^k\|^2 \|D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma)\|^2 ds \le \\
&\le \|\Sigma\|^2 \left[\mathbb{E} \int_0^t \|\bar{S}_u^k\|^4 ds \right]^{1/2} \left[\mathbb{E} \int_0^t \|D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma)\|^4 ds \right]^{1/2}
\end{aligned}$$

Now we use the boundedness of $\mathbb{E}[\|\bar{S}_u^k\|^4]$ to get

$$\mathbb{E}[|(i)|^2] \le M \left[\mathbb{E} \int_0^t \|D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma)\|^4 ds \right]^{1/2}$$

for suitable $M > 0$. At this point we use the Hypothesis 13-(ii) and the independence od $\mathcal{F}$ and $\overline{W}$. In fact, taking conditional expectation:

$$\begin{aligned}
&\mathbb{E} \int_0^t \|D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma)\|^4 \; ds = \\
&= \int_0^t \mathbb{E} \left[\mathbb{E} \left[\|D_s C_k(u, S_u^k) - D_s C_k(u, S_u^\sigma)\|^4 | \overline{W}_u \right] \right] \; ds \\
&= \int_0^t \mathbb{E} \left[\int_{\mathbb{R}_+^n} \|D_s C_k(u, s\mathcal{E}(\varepsilon_k \overline{W}_u)) - D_s C_k(u, s)\|^4 \rho_{S_u^\sigma}(s) ds \right] ds.
\end{aligned}$$

Now we recall that the sequence $(D_s C_k)_{k \in \mathbb{N}}$ converges a.e. in $[0,T] \times \mathbb{R}_+^n$ and it is bounded, so by the dominated convergence theorem it is compact in the space $L^p([0,T] \times \mathbb{R}_+^n)$ with weight $\rho_{S_u^\sigma}(s)$. It follows that, by the change of variable $s_i = e^{z_i}$ also the sequence $(D_s C_k(\cdot, e^{\cdot}))_{k \in \mathbb{N}}$ is compact in the space $L^p([0,T] \times \mathbb{R}^n)$ with weight $\rho_{S_u^\sigma}(e^z) e^z$. We can then apply the Fréchet-Kolmogorov Theorem (see [33, p. 275], where only the case of $L^p(\mathbb{R})$ without weight is considered: however the proof can be easily extended to our weightd spaces) to our case getting

$$\begin{aligned}
&\int_0^t \mathbb{E} \left[\int_{\mathbb{R}_+^n} \|D_s C_k(u, s\mathcal{E}(\varepsilon_k \overline{W}_u)) - D_s C_k(u, s)\|^4 \rho_{S_u^\sigma}(s) ds \right] du = \\
&= \mathbb{E} \int_0^t \left[\int_{\mathbb{R}^n} \|D_s C_k(u, e^{z + \varepsilon_k \overline{W}_u - \varepsilon_k^2 u/2} - D_s C_k(u, e^z)\|^4 \rho_{S_u^\sigma}(e^z) e^z dz \right] du \le \\
&\le \mathbb{E} \int_0^t \omega \left(|\varepsilon_k \overline{W}_u - \varepsilon_k^2 u/2| \right) \; du \overset{k \to +\infty}{\longrightarrow} 0
\end{aligned}$$

(here ω denotes a modulus). For the second term we have

$$
\begin{aligned}
\mathbb{E}[|(ii)|^2] &\le \|\Sigma\|^2 \mathbb{E}\int_0^t \|\bar{S}^\sigma_u\|^2 \|D_s C_k(u,S^\sigma_u) - D_s C(u,S^\sigma_u)\|^2 \, ds \le \\
&\le \|\Sigma\|^2 \left[\mathbb{E}\int_0^t \|\bar{S}^\sigma_u\|^4 ds\right]^{1/2} \left[\mathbb{E}\int_0^t \|D_s C_k(u,S^\sigma_u) - D_s C(u,S^\sigma_u)\|^4 ds\right]^{1/2} \le \\
&\le M \left[\mathbb{E}\int_0^t \|D_s C_k(u,S^\sigma_u) - D_s C(u,S^\sigma_u)\|^4 ds\right]^{1/2}
\end{aligned}
$$

for suitable $M > 0$. Then, using again Hypothesis 13-(ii) we get

$$
\mathbb{E}\int_0^t \|D_s C_k(u,S^\sigma_u) - D_s C(u,S^\sigma_u)\|^4 ds = \int_0^t \int_{\mathbb{R}^n_+} \|D_s C_k(u,s) - D_s C(u,s)\|^4 \rho_{S^\sigma_u}(s) ds du
$$

which goes to 0 as $k \to +\infty$ thanks to the dominated convergence theorem.

Finally we have, for the last two terms:

$$
\begin{aligned}
\mathbb{E}[|(iii)|^2] &= \mathbb{E}\left[\int_0^t \|D_s C(u,S^\sigma_u)\|^2 |S^k_u - S^\sigma_u|^2 \|\sigma_u\|^2 \, du\right] \le \\
&\le \|\Sigma\|^2 \|h\|^2_{\mathrm{Lip}} \int_0^t \mathbb{E}\left[|S^\sigma_u|^2 (1 - \mathcal{E}(\varepsilon_k \bar{W}))^2\right] \le \\
&\le \|\Sigma\|^2 \|h\|^2_{\mathrm{Lip}} t \left(\mathbb{E}[|S_u|^4]\right)^{1/2} \left(\mathbb{E}[|1 - \mathcal{E}(\varepsilon_k \bar{W})|^4]\right)^{1/2}
\end{aligned}
$$

where $\|\Sigma\| = \sup_{\gamma\in\Sigma} \|\gamma\|$. The last term tends to 0 as $k \to \infty$, so $\mathbb{E}[|(iii)|^2] \to 0$. Then:

$$
\begin{aligned}
\mathbb{E}[|(iv)|^2] &= \varepsilon_k^2 \mathbb{E}\left[\int_0^t \|D_s C_k(u,S^k_u)\|^2 \|S^k_u\|^2 \, du\right] \le \\
&\le \varepsilon_k^2 \|h\|^2_{\mathrm{Lip}} \mathbb{E}[t\|S^k_\cdot\|^2_\infty] \le \varepsilon_k^2 t \|h\|^2_{\mathrm{Lip}} B_2 (1 + |s|^2)
\end{aligned}
$$

so $\mathbb{E}[|(iv)|^2] \to 0$ as $k \to \infty$.

To conclude we have obtained that for all $t \ge 0$

$$
\lim_{k\to\infty} \mathbb{E}[|\Pi^k_t - \Pi_t|^2] = 0
$$

Since $\Pi^k - \Pi$ is a martingale, we also have, thanks to Doob's inequality, that

$$
\lim_{k\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |\Pi^k_t - \Pi_t|^2\right] = 0
$$

This means that, on a suitable subsequence $\lim_{k\to\infty} \Pi^k_t = \Pi_t$ almost surely, and the limit is uniform in t. $\square$

Corollary 20 *If Hypothesis 13 holds and C denotes the unique viscosity solution of Eq. (3), then $\Delta_t = D_s C(t,S_t)$ is a.s. well defined and $(C_t, \Delta_t)_t$ is a superstrategy, where $C_t = C(t,S_t)$.*

Proof. Since $C \in W^{1,p}_{\mathrm{loc}}$, then $\Delta_t = D_s C(t,S_t)$ is a.s. well defined. Besides, we have that $e_t = \Pi_t - C_t$. Since $e = \lim_{k\to\infty}(\Pi^k - C^k)$ a.s. uniformly in t, and $\Pi^k - C^k$ are increasing processes, then also e_t is, and we have the result. $\square$

5 Examples and Applications

Now we present some examples. Their order is not casual: in fact they use the theory presented above in a progressive way. More precisely, the first example shows the results via an explicit formula; in the second one, Σ is contained in $\mathrm{GL}(n,\mathbb{R})$, so the solution of the BSB equation is smooth; in the third one, Σ is not contained in $\mathrm{GL}(n,\mathbb{R})$ but Hypothesis 13 holds and we can apply the results of Section 4; finally in the fourth one we can not apply the theoretical results of this article but the BSB has an explicit smooth solution and the BSB strategy is superhedging.

5.1 Margrabe's exchange option

We consider Margrabe's exchange option (see [26]), whose final payoff is the following:

$$h(s) = (s_1 - \lambda s_2)^+$$

We notice that h is convex, so we can apply the results of the previous section if Hypothesis 13 (ii) holds. In this case we can check directly the correctness of this: in fact, Hypothesis 13 (ii) implies that $\Gamma = \max_{\gamma\in\Sigma} \|\gamma_1 - \gamma_2\|_{\mathbb{R}^2} > 0$, so there exists an explicit solution to the BSB equation, given by

$$C(t, s_1, s_2) = s_1 N(d_1) - \lambda s_2 N(d_2) \tag{12}$$

where

$$d_1 = \frac{1}{\Gamma\sqrt{T-t}} \ln\left(\frac{s_1}{\lambda s_2}\right) + \frac{1}{2}\Gamma\sqrt{T-t}, \quad d_2 = d_1 - \Gamma\sqrt{T-t}$$

and N is the cumulative distribution function of a centered reduced Gaussian random variable:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}}\, dx$$

(see [27] for details). Since $C \in C^{1,2}$, we can apply Theorem 3 directly.

Margrabe's exchange option is a rare case in which there is an explicit solution to the BSB equation. In the general case we are not so lucky and we have to use the theory of Sections 3 and 4.

5.2 RiskMetrics' stochastic volatility

This example is a generalisation of a stochastic volatility model proposed in [34]. We consider a model in which the volatility is

$$\sigma_t = \sigma_0 \mathbf{1}_{t<t_0} + \sum_{i=1}^{k} \sigma_i \mathbf{1}_{t_{i-1}\le t<t_i} + \sigma_{t_k} \mathbf{1}_{t_k \le t}$$

where t_i are fixed instants in $[0,T]$ and $\sigma_1, \ldots, \sigma_k$ are discrete random variables taking values in $\mathrm{GL}(n,\mathbb{R})$ independent of W and such that σ_i is $\mathcal{F}_{t_i}$-measurable for all $i = 1, \ldots, k$. With these assumptions, Hypothesis 13 (ii) holds. Moreover, since σ assumes values in $\mathrm{GL}(n,\mathbb{R})$, thanks to the results of Section 3, the BSB equation has a smooth solution for all final payoff continuous and with linear growth.

As we have seen in Example 11, it is not always the case that Σ is contained in $\mathrm{GL}(n,\mathbb{R})$, as the next example shows.

5.3 Markov stochastic correlation

We consider a model in which there are two assets driven by the 2-dimensional Brownian motion (W^1, W^2) and Σ is as in Example 11. Moreover we assume that the correlation ρ is given by $\rho_t = \frac{2}{\pi} \arctan Z_t$, where Z is an Ornstein-Uhlenbeck process independent of (W^1, W^2) having the dynamics

$$dZ_t = (a - bZ_t)\ dt + \sigma_3\ dW_t^3$$

with W^3 independent of (W^1, W^2). Then Z_t has a Gaussian law $N(\mu_{Z_t}, \sigma^2_{Z_t})$, where $\mu_{Z_t} = e^{-bt}Z_0 + a(1-e^{-bt})/b$ and $\sigma^2_{Z_t} = \sigma_3^2(1-e^{-2bt})/2b$. In this model, ρ assumes values in $(-1, 1)$ and the BSB equation is not uniformly parabolic. However we can use the results of Section 4 if we prove that Hypothesis 13 (ii) holds. This is equivalent to prove that the random variables

$$\begin{aligned} Y_t &= \frac{1}{\sigma_1}\left(\log\frac{S_t}{S_0} + \frac{1}{2}\sigma_1^2 t\right) = W_t^1 \\ Y_t &= \frac{1}{\sigma_2}\left(\log\frac{S_t}{S_0} + \frac{1}{2}\sigma_2^2 t\right) = \rho_t W_t^1 + \sqrt{1-\rho_t^2}W_t^2 \end{aligned}$$

has a joint density with respect to the 2-dimensional Lebesgue measure. In order to see this, we take a generic $f \in C_b(\mathbb{R}^2)$ and calculate

$$\begin{aligned} &\mathbb{E}[f(Y_t^1, Y_t^2)] = \mathbb{E}[f(W_t^1, \rho_t W_t^1 + \sqrt{1-\rho_t^2}W_t^2)] = \\ &= \int_{\mathbb{R}^3} f\left(x_1, \frac{2}{\pi}x_1 \arctan x_3 + x_2\sqrt{1-\left(\frac{2}{\pi}\arctan x_3\right)^2}\right) e^{-\frac{x_1^2+x_2^2}{2t} - \frac{(x_3-\mu_{Z_t})^2}{2\sigma^2_{Z_t}}}\ dx \end{aligned}$$

By making the change of variable

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= \frac{2}{\pi}x_1 \arctan x_3 + x_2\sqrt{1-\left(\frac{2}{\pi}\arctan x_3\right)^2} \\ y_3 &= x_3 \end{aligned}$$

which is C^1 and invertible, we arrive to

$$\mathbb{E}[f(Y_t^1, Y_t^2)] = \int_{\mathbb{R}^3} f(y_1, y_2)\phi(y_1, y_2)\ dy$$

where ϕ is defined by

$$\begin{aligned} \phi(y_1, y_2) &= \int_{\mathbb{R}} \exp\left(-\frac{y_1^2}{2t} - \frac{1}{2t}\left(\frac{y_2 - \frac{2}{\pi}y_1 \arctan x_3}{\sqrt{1-\left(\frac{2}{\pi}\arctan y_3\right)^2}}\right)^2 - \frac{(x_3-\mu_{Z_t})^2}{2\sigma^2_{Z_t}}\right) \times \\ &\quad \times \left(1 - \left(\frac{2}{\pi}\arctan y_3\right)^2\right)^{-1/2}\ dy_3 \end{aligned}$$

In particular we obtained that (Y^1, Y^2) has a density with respect to the Lebesgue measure, so Hypothesis 13 (ii) holds. This means that we can apply the results of Section 4 to every payoff h satisfying Hypothesis 13 (i) (to cite some examples, call options on the maximum of several assets and call and put options on the arithmetic mean).

The last example shows that the picture is still not complete. In fact, even BSB equations which are not uniformly parabolic and with non-convex final condition can have smooth solutions, as the next example shows.

5.4 An example of non-convex payoff

We consider an option whose final payoff is the following:

$$h(s) = (\sqrt{s_1 s_2} - K)^+$$

and a model in which Σ is as in Example 11. In this case, Σ is not contained in $GL(n, \mathbb{R})$, so we can not use the results in Section 3 and conclude that the BSB equation has a smooth solution. Moreover h is not convex, so neither we can apply the results of Section 4. However in this case there exists an explicit smooth solution to the BSB equation, given by

$$C(t, s_1, s_2) = e^{\mu(T-t)} \sqrt{s_1 s_2} N(d_1) - K N(d_2) \tag{13}$$

where $\mu = -\frac{1}{8}(\sigma_1 - \sigma_2)^2$ where

$$d_1 = \frac{1}{\tilde{\gamma}_t} \ln \frac{\sqrt{S_1(t) S_2(t)}}{K e^{-\mu(T-t)}} + \frac{1}{2}\tilde{\gamma}_t, \quad d_2 = d_1 - \tilde{\gamma}_t, \quad \tilde{\gamma}_t = \frac{1}{2}|\sigma_1 + \sigma_2|\sqrt{T-t}$$

(see [27] for details).

A The BSB equation: existence, uniqueness and regularity

The BSB equation (3) (and also its transform (8)) is a fully nonlinear parabolic equation and belongs to the more general class of the so-called Hamilton-Jacobi-Bellman equations (HJB). By simplifying the subject, we can say that there are two main approaches to study this kind of equations.

The first one is to look for solutions with the derivatives taken in the classical or in the Sobolev sense in spaces $C^{2,\alpha}$ or in spaces $W^{2,p}$; the second one is to look for viscosity solutions in the sense introduced by Crandall and Lions (see [5]).

Looking at the classical solution approach we recall first the work of Krylov (see e.g. Krylov's book [20]) where classical solution of HJB equations are given by assuming that the initial datum $h \in C_b^3(\mathbb{R}^n)$ (see more precisely [14] p. 169, [20] p. 301). In Krylov's approach the solution inherits its regularity from the initial datum h. This is not good for our problem since the typical h characterizing a European asset is C^0 and with linear growth. Krylov's results could be somehow used in our problem when the datum h is semiconvex or semiconcave since in this case one could get semiconvexity or semiconcavity of the solutions. However such a result is not known at the moment for our equation. Moreover, this theory is quite difficult to handle for non-specialists.

The other important issue in the classical solution approach is the one of giving interior regularity estimates for the parabolic equations, a method introduced by Caffarelli and then studied many others (see e.g. [7, 30] and the references quoted therein). This methods gives interior regularity of the solution of the parabolic equation without assuming regularity of the initial datum and is therefore good for our problem. This method, however, applies in its best performance only in the uniformly parabolic case. In this case then (see Section 3) we can get $C^{2,\alpha}$ solutions and apply Theorem 3, while in the other case we are forced to use or the Krylov approach or the viscosity solution approach recalled below.

The other dominant approach in dealing with nonlinear PDEs is the theory of viscosity solutions. This theory, developed by Crandall and Lions in a series of papers in the early '80s, gives general existence and uniqueness theorems, but the solutions are in general only continuous, and to prove regularity one has to use one of the approaches recalled above. So if we want to apply Theorem 3 with a viscosity solution, we also need to prove that the solution is smooth enough to apply the Itô formula.

Moreover, there are interesting links between optimal control problems and viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations (see section 2): in particular, under general assumptions that are satisfied in our case, the value function is the unique viscosity solution of the HJB equation. The interested reader can see [6] for a detailed introduction to the concept of viscosity solutions and its applications, and [14] for the links between optimal stochastic control and viscosity solutions of HJB equations.

In the following we will show what kind of results we can get on our BSB equation which is, as explained in Section 2, a special case of a HJB equation. We will consider a family of HJB equations (related to optimal control problems) that is more general than the one we study. For this family of HJB equations we show that in the general case there exists a unique viscosity solution with the space derivative defined almost everywhere and belonging to L^p_{loc}, provided the final datum is polynomially growing and locally Lipschitz. In the special case when uniform parabolicity holds we have shown (Theorem 10 in Section 3) that the viscosity solution is in fact more regular, i.e. $C^{2,\alpha}$ (once differentiable in time and twice in space with Hölder continuous derivatives).

We point out that, while the regularity results for the uniformly parabolic case can be found in the literature (see [30] and [7]), to our knowledge, the results on viscosity solutions stated above are not given in the current literature. For this reason we will give here a complete proof of them. Such a proof uses in fact standard ideas from the theory of viscosity solution (that can be found e.g. in [23, 24, 16, 17, 14]) and one could say that the results we prove were clearly expected. However, due to the important applications of the BSB equation, we think it is useful to give a detailed statement and proof of our existence, uniqueness and regularity results.

To keep the lenght of this appendix shorter we will use the notations and follow the approach of Chapter V of [14], which is a well known reference on the topic, by underlying the points that does not fit in their assumptions and giving a complete proof of them. We also observe that the set of assumptions we use is not always sharp, but is enough for our need.

A.1 The notion of viscosity solutions

We recall here the notion of viscosity solution of an equation of the kind

$$-\frac{\partial}{\partial t}V(t,x) + F(t,x,D_xV(t,x),D_x^2V(t,x),V(t,x)) = 0 \tag{14}$$

where $(t,x) \in Q = [0,T) \times O$, O is an open subset of $\mathbb{R}^n$, and F is a continuous function such that

$$F(t,x,p,A+B,V) \leq F(t,x,p,A,V)$$

for all $(t,x) \in Q$, $p \in \mathbb{R}^n$, $V \in \mathbb{R}$ and $A, B \in \mathcal{S}^n$ with $B \geq 0$, where $\mathcal{S}^n$ is the set of all $n \times n$ symmetric matrices.

We say that a function $V \in C([0,T] \times O)$

- is a **viscosity subsolution** of Eq. (14) if for each $v \in C^{1,2}(Q)$,

$$-\frac{\partial}{\partial t} v(\bar{t},\bar{x}) + F(\bar{t},\bar{x}, D_x v(\bar{t},\bar{x}), D_x^2 v(\bar{t},\bar{x}), v(\bar{t},\bar{x})) \leq 0$$

  at every $(\bar{t},\bar{x}) \in Q$ which is a local maximum of $V - v$ on Q, with $V(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$.

- is a **viscosity supersolution** of Eq. (14) if for each $v \in C^{1,2}(Q)$,

$$-\frac{\partial}{\partial t} v(\bar{t},\bar{x}) + F(\bar{t},\bar{x}, D_x v(\bar{t},\bar{x}), D_x^2 v(\bar{t},\bar{x}), v(\bar{t},\bar{x})) \geq 0$$

  at every $(\bar{t},\bar{x}) \in Q$ which is a local minimum of $V - v$ on Q, with $V(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$.

- V is a **viscosity solution** of Eq. (14) if it is both a viscosity subsolution and a viscosity supersolution of Eq. (14) in Q.

A viscosity solution is a solution in a weaker sense than the classical one, but it turns out that is is the "right" definition in the area of stochastic optimal control theory. In fact we have the following lemma (see [14] pag. 69):

Lemma 21 *If $V \in C^{1,2}(Q)$, then V is a viscosity solution of Eq. (14) in Q if and only if it is a classical solution of Eq. (14) in Q.*

A.2 An existence theorem

Now we recall how the notion of viscosity solution can be linked to a stochastic optimal control problem. In fact, stochastic optimal control provide a nice tool for the existence theorem of a nonlinear second order equation of the kind (14) in the case when F is convex.

We consider the following control problem. We suppose that X is a $\mathbb{R}^n$-valued process governed by the stochastic differential equation

$$dX_t = f(t, X_t, u_t)\; dt + \sigma(t, X_t, u_t)\; dW_t, \quad t \in [0,T] \tag{15}$$

where u is a progressively measurable process taking values in a closed set $U \subseteq \mathbb{R}^m$, and W is a d-dimensional Brownian motion. Moreover, we fix $Q_0 = [0,T) \times \mathbb{R}^n$, $\bar{Q}_0$ the closure of Q_0, and we make the assumptions that f and σ are continuous on $\bar{Q}_0 \times U$, $f(\cdot,\cdot,v)$, $\sigma(\cdot,\cdot,v)$ are of class $C^1(\bar{Q}_0)$, and for some constant C

$$\begin{aligned} &\text{(a)} \quad |f_t| + |D_x f| \leq C, \qquad |\sigma_t| + |D_x \sigma| \leq C, \\ &\text{(b)} \quad |f(t,x,v)| \leq C(1 + |x| + |v|), \\ &\qquad\;\; |\sigma(t,x,v)| \leq C(1 + |x| + |v|) \end{aligned} \tag{16}$$

We want to choose the control u to minimize

$$J(t,x;u) = \mathbb{E}_{t,x}\left[\int_t^T L(s, X_s, u_s)\; ds + \psi(X_T)\right]$$

where L, ψ and q are continuous functions such that

$$\begin{array}{ll}\text{(a)} & |L(t,x,v)| \leq C(1+|x|^k+|v|^k) \\ \text{(b)} & |\psi(t,x)| \leq C(1+|x|^k)\end{array} \tag{17}$$

for suitable constants $C > 0$, $k \geq 0$.

Let us restate the problem in a more systematic way. We call **reference probability system** ν a 4-uple

$$\nu = (\Omega, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{P}, W)$$

where $(\Omega, \mathcal{F}_T, \mathbb{P})$ is a probability space, and W is a Brownian motion adapted to the filtration $(\mathcal{F}_s)_s$. We denote with

$$\mathcal{A}_{t,\nu} = \{u \mid u \text{ progressively measurable } U\text{-valued process defined on } \nu\}$$

We consider the infimum of J among all $u \in \mathcal{A}_{t,\nu}$:

$$V_\nu(t,x) = \inf_{u\in\mathcal{A}_{t,\nu}} J(t,x;u)$$

and we call **value function** the function $V(t,x)$ defined as

$$V(t,x) = \inf_\nu V_\nu(t,x)$$

This problem is linked to the following equation, called **dynamic programming equation** or **Hamilton-Jacobi-Bellman** (HJB) equation:

$$-\frac{\partial V}{\partial t}(t,x) + \mathcal{H}(t,x,D_xV(t,x),D_x^2V(t,x)) = 0, \quad (t,x) \in Q_0, \tag{18}$$

with boundary condition

$$V(T,x) = \psi(x) \quad \forall x \in \mathbb{R}^n$$

where

$$\mathcal{H}(t,x,p,A) = \sup_{v\in U}\left(-\langle f(t,x,v),p\rangle - \frac{1}{2}\text{tr } Aa(t,x,v) - L(t,x,v)\right)$$

and $a = \sigma\sigma^*$. We notice that this equation is of the general kind (14) with f replaced by $\mathcal{H}$, and that the BSB equation is a HJB equation with $a(t,x,v) = (\bar{x}v)(\bar{x}v)^*$, $f = L = q = 0$. Moreover we can extend the function h defined on $\mathbb{R}^n_+$ to a function $\psi \in C_p(\mathbb{R}^n)$ in such a way that $h = \psi$ on $\mathbb{R}^n_+$.

One important tool in proving that V is a solution of Eq. (18) is the following. We say that a function W has **property (DP)** (dynamic programming) if for every ν, every control $u \in \mathcal{A}_{t,\nu}$ and every stopping time θ taking values in $[t,T]$ we have

$$W(t,x) \leq \mathbb{E}_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\ ds + W(\theta,X_\theta)\right]$$

and for every $\delta > 0$, there exists a ν and a control $u \in \mathcal{A}_{t,\nu}$ such that

$$W(t,x) + \delta \geq \mathbb{E}_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\ ds + W(\theta,X_\theta)\right]$$

for every stopping time θ taking values in $[t,T]$.

Usually the property (DP) is stated for the value function V. Here we are stating it for a generic function, because in the proof of existence of a viscosity solution of Eq. (18) we will build value functions of auxiliary control problems for which property (DP) will hold.

Theorem 22 *If property (DP) holds for V (the assumptions above are satisfied) and $V \in C(Q_0)$, then V is a viscosity solution of Eq. (18) in Q_0.*

Proof. See [14] pag. 69. $\square$

We can see that if property (DP) holds for the value function V and V is continuous, then V is a viscosity solution of Eq. (18), so we are led to study these facts. To this purpose, we assume that

$$\begin{array}{ll} \text{(a)} & U \text{ is compact} \\ \text{(b)} & f,\, \sigma \text{ are continuous on } \bar{Q}_0 \times U, \text{ and } f(\cdot,\cdot,v),\, \sigma(\cdot,\cdot,v) \\ & \text{are of class } C^1(\bar{Q}_0) \text{ for each } v \in U \\ \text{(c)} & \text{for suitable } C_1, C_2 > 0 \\ & |f_t| + |D_x f| \le C_1, |\sigma_t| + |D_x \sigma| \le C_1 \\ & |f(t,0,v)| + |\sigma(t,0,v)| \le C_2 \end{array} \tag{19}$$

and that

$$|L(t,x,v)| \le C_3(1+|x|^k) \tag{20}$$

The standard results concerning existence and uniqueness of viscosity solutions require that the final payoff ψ is bounded (some papers like [22] and [32] deal with the case of unbounded ψ but their assumptions does not include our BSB equation). For many financial applications this is not sufficient. Now we prove an existence theorem for ψ unbounded, which extends results already existing in [14].

We say that ψ and its derivatives until the k-th one have **polynomial growth**, and we indicate it with $\psi \in C_p^m(\mathbb{R}^n)$, if for all $i = 0, \cdots, m$ there exist C, k such that

$$|\psi^{(i)}(x)| \le C(1+|x|^k) \quad \forall x \in \mathbb{R}^n$$

Theorem 23 (Existence) *If assumptions (19) and (20) hold, and $\psi \in C_p(\mathbb{R}^n)$, then V is a viscosity solution of Eq. (18) in Q_0.*

In the proof we use the following results.

Proposition 24 *For all $u \in \mathcal{A}_{t,\nu}$, $m > 0$, we have*

$$\mathbb{E}_{t,x}[\|X.\|_\infty^m] \le B_m(1+|x|^m)$$

where the norm $\|\cdot\|$ is the sup norm in $C([t,T])$, and the constant B_m depends only on T and the constant C in assumptions (16).

Proof. See [14] pagg. 398–399 and [19] pag. 85. $\square$

From this result, we get that for $M > 0$,

$$\mathbb{P}\{\|X.\|_\infty \ge M\} \le \frac{1}{M}\mathbb{E}[\|X.\|_\infty] \le \frac{B_1}{M}(1+|x|)$$

Besides, by definition of V, if L and ψ have polynomial growth, then

$$|V(t,x)| \le M(T-t)(1+|x|^k)$$

where the constant M depends only on C_1, C_2 and C_3 in assumptions (19) and (20).

Theorem 25 *If assumptions (19) and (20) hold, then V is continuous on $\bar{Q}_0$, and property (DP) holds. Moreover $V = V_\nu$ for every reference probability system ν.*

Proof. If $\psi = 0$, the proof follows from [14] pages 185–188. If $\psi \in C_p^2(\mathbb{R}^n)$, then by Dynkin's formula (see[14] p. 178) we have

$$\mathbb{E}_{t,x}[\psi(X_T)] = \psi(x) + \mathbb{E}_{t,x}\left[\int_t^T G^u\psi(X_s)\ ds\right]$$

A consequence is that

$$J(t,x;u) = \psi(x) + \mathbb{E}_{t,x}\left[\int_t^T \tilde{L}(s, X_s, u_s)\ ds\right]$$

where $\tilde{L} = L + G^u\psi$. Since $\psi(x)$ is not affected by the control u, the problem of maximizing J is equivalent to maximize the last term in the right hand side. We also notice that the function $\tilde{L}$ still satisfies assumption (20), so we can build another control problem with $\tilde{L}$ instead of L and final condition $\psi = 0$, so we are in the previous situation and the theorem holds. $\square$

Lemma 26 *If $\psi \in C_p(\mathbb{R}^n)$, then there exists a sequence $(\psi_m)_m$ in $C_p^2(\mathbb{R}^n)$ such that $\psi_m \to \psi$ uniformly on compact sets. Moreover, there exist C, k such that $|\psi_m(x)| \le C(1 + |x|^k)$ uniformly with respect to m.*

Proof. For $m = 1, 2, \ldots$, let $\rho_m \in C^\infty(\mathbb{R}^n)$ be such that $\rho_m \ge 0$, $\int_{\mathbb{R}^n} \rho_m(x)\ dx = 1$ and $\rho_m(x) = 0$ if $|x| \ge 1/m$. We let

$$\psi_m(x) = (\psi * \rho_m)(x) = \int_{\mathbb{R}^n} \psi(y)\rho_m(x-y)\ dy$$

Then $\psi_m \in C^\infty(\mathbb{R}^n)$. Besides, since ψ is uniformly continuous on each compact set K, for all $\varepsilon > 0$ there exists δ (which depends on ε and K) such that

$$|\psi(x-y) - \psi(x)| < \varepsilon \quad \forall x \in K, |y| \le \delta$$

Then for all $x \in K$, $m > 1/\delta$:

$$\begin{aligned} |\psi_m(x) - \psi(x)| &= \left|\int_{\mathbb{R}^n} (\psi(x-y) - \psi(x))\rho_m(y)\ dy\right| \le \\ &\le \int_{|y|\le 1/m} |\psi(x-y) - \psi(x)|\rho_m(y)\ dy \le \varepsilon \int_{|y|\le 1/m} \rho_m(y)\ dy = \varepsilon \end{aligned}$$

so $\psi_m \to \psi$ uniformly on compact sets. Besides:

$$\begin{aligned} |\psi_m(x)| &\le \int_{\mathbb{R}^n} \psi(y)\rho_m(x-y)\ dy \le \int_{|y|\le 1/m} |\psi(x-y)|\rho_m(y)\ dy \le \\ &\le \int_{|y|\le 1/m} C\left(1 + \left(|x| + \frac{1}{m}\right)^k\right)\rho_m(y)\ dy = \\ &= C\left(1 + \left(|x| + \frac{1}{m}\right)^k\right)\int_{|y|\le 1/m} \rho_m(y)\ dy \le C\left(1 + (|x|+1)^k\right) \end{aligned}$$

so ψ_m has polynomial growth and the uniform estimate holds. $\square$

Now we prove our theorem.

Proof. If $\psi \in C_p^2(\mathbb{R}^n)$, then we get the result by Theorem 25 and Theorem 22. If $\psi \in C_p(\mathbb{R}^n)$, then by the previous lemma there exists a sequence $(\psi_m)_m$ in $C_p^2(\mathbb{R}^n)$ such that $\psi_m \to \psi$ uniformly on compact sets. Let $V_{m,\nu}, V_m$ be the corresponding value functions. By definition of V and V_m, for each $\delta > 0$ there exists ν and $u \in \mathcal{A}_{t,\nu}$ such that

$$\begin{aligned}
& V_m(x) - V(x) + \delta \\
\leq\ & E_{t,x}\left[\int L(s,X_s,u_s)\ ds + \psi_m(X_T)\right] - V(x) \\
\leq\ & \mathbb{E}_{t,x}\left[\int L(s,X_s,u_s)\ ds + \psi_m(X_T)\right] - \mathbb{E}_{t,x}\left[\int L(s,X_s,u_s)\ ds + \psi(X_T)\right] = \\
=\ & \mathbb{E}_{t,x}\left[\psi_m(X_T) - \psi(X_T)\right] = \\
=\ & \mathbb{E}_{t,x}\left[(\mathbf{1}_{|X_T|\leq M} + \mathbf{1}_{|X_T|>M})(\psi_m(X_T) - \psi(X_T))\right] = \\
\leq\ & \|\psi_m - \psi\|_{B(0,M)} + \mathbb{E}_{t,x}\left[\mathbf{1}_{|X_T|>M}(\psi_m(X_T) - \psi(X_T))\right] = I_1 + I_2
\end{aligned}$$

where $\|\cdot\|_{B(0,M)}$ denotes the sup norm in $B(0,M)$. An analogous estimation holds for $V(x) - V_m(x) + \delta$. Since ψ_m and ψ have polynomial growth, we have

$$\begin{aligned}
I_2 \ &\leq\ \|\mathbb{E}_{t,x}\left[2C(1+|X_T|^k)\mathbf{1}_{|X_T|>M}\right] \leq \\
&\leq\ \left(\mathbb{P}\{|X_T| > M\}\mathbb{E}_{t,x}\left[4C^2(1+|X_T|^k)^2\right]\right)^{1/2} \leq \\
&\leq\ \left(\mathbb{P}\{\|X_\cdot\|_\infty > M\}C_1\left(1 + \mathbb{E}_{t,x}[\|X_\cdot\|_\infty^{2k}]\right)\right)^{1/2} \leq \\
&\leq\ \frac{B_1}{M}(1+|x|)C_1(1+|x|^{2k})^{1/2} \leq \frac{C_2}{M}(1+|x|^{k+1})
\end{aligned}$$

So I_2 can be made arbitrarily small by choosing a suitable M and x in a given compact set. Besides I_1 converges to 0 as $m \to \infty$, so we have that $V_m \to V$ uniformly on compact sets, so V is continuous. Now let's prove that property (DP) holds for V. In order to prove the second inequality, for an arbitrary stopping time θ we calculate

$$\begin{aligned}
& \left|E_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\ ds + V(\theta,X_\theta) - \int_t^\theta L(s,X_s,u_s)\ ds + V_m(\theta,X_\theta)\right]\right| \\
=\ & |\mathbb{E}_{t,x}\left[V(\theta,X_\theta) - V_m(\theta,X_\theta)\right]| \\
\leq\ & \mathbb{E}_{t,x}\left[(\mathbf{1}_{|X_\theta|\leq M} + \mathbf{1}_{|X_\theta|>M})|V(X_\theta) - V_m(X_\theta)|\right] \\
\leq\ & \|V - V_m\|_{B(0,M)} + \mathbb{E}_{t,x}\left[\mathbf{1}_{|X_\theta|>M}|V(X_\theta) - V_m(X_\theta)|\right] = I_3 + I_4
\end{aligned}$$

By Lemma 26 we have

$$\begin{aligned}
I_4 \ &\leq\ \mathbb{E}_{t,x}\left[\mathbf{1}_{|X_\theta|>M}2C(1+|X_\theta|^k)\right] \leq \\
&\leq\ \left(\mathbb{P}\{|X_\theta| > M\}\mathbb{E}_{t,x}\left[4C^2(1+|X_\theta|^k)^2\right]\right)^{1/2} \leq \\
&\leq\ \left(\mathbb{P}\{\|X_\cdot\|_\infty > M\}C_1\left(1 + \mathbb{E}_{t,x}[\|X_\cdot\|_\infty^{2k}]\right)\right)^{1/2} \leq \frac{C_2}{M}(1+|x|^{k+1})
\end{aligned}$$

Also here I_4 can be made arbitrarily small by choosing a suitable M. In particular, for each $\delta > 0$ there exists M such that $I_4 < \delta/3$. Moreover, there exists m such that $I_3 < \delta/3$. At last, by Theorem 22, property (DP) holds for V_m, so for each $\delta > 0$, there exists a ν and a control $u \in \mathcal{A}_{t,\nu}$ such that

$$V_m(t,x) + \frac{\delta}{3} \geq \mathbb{E}_{t,x}\left[\int_t^\theta L(s, X_s, u_s)\ ds + V_m(\theta, X_\theta)\right]$$

for every stopping time θ taking values in $[t,T]$. By putting together these three inequalities, we obtain property (DP) for V, so the theorem follows from Theorems 25 and 22. □

Corollary 27 *If $\psi \in C_p(\mathbb{R}^n)$, then the value function V_ψ defined in Equation (7) is a viscosity solution of Eq. (3) in $[0,T] \times \mathbb{R}^n_+$. Besides, if $\psi, \psi' \in C_p(\mathbb{R}^n)$ are such that $\psi = \psi'$ in $(\mathbb{R}^+)^n$, then $V_\psi = V_{\psi'}$ in $[0,T] \times (\mathbb{R}^+)^n$.*

Proof. The controlled process S satisfies assumptions (19) and (20), so by Theorem 23 the value function V_ψ is a viscosity solution of Eq. (3) in the whole space $[0,T] \times \mathbb{R}^n$, so it is a viscosity solution also in the subset $[0,T] \times \mathbb{R}^n_+$. Since the initial condition $s^i > 0$ implies $S^i_T > 0$ $\mathbb{Q}$-a.s., then the second part of the theorem follows by the definition of V_ψ. □

A.3 A uniqueness theorem

Now we cite a comparison result by Lions [24] that will imply a uniqueness theorem for Equation (18). In order to obtain the comparison result, we will assume, besides (16), (19) and (20), that

$$\begin{array}{ll} \text{(a)} & \sup_{u \in U, t \in [0,T]} \sum_{i,j=1}^n \|\sigma_{ij}(t,\cdot,u)\|_{W^{2,\infty}(\bar{B}_R)} + \|f_i(t,\cdot,u)\|_{W^{1,\infty}(\bar{B}_R)} < +\infty \quad \forall R < +\infty \\ \text{(b)} & q, L \text{ are bounded continuous in } \bar{B}_R \text{ uniformly in } u \in U,\ t \in [0,T]\ \forall R < +\infty \\ \text{(c)} & \sigma_{ij}, f_i, q, L \text{ are continuous in } u \text{ for all } (t,x) \in [0,T] \times \mathbb{R}^n \\ \text{(d)} & q, L \text{ are continuous in } t \text{ uniformly in } u \in U,\ x \in \mathbb{R}^n \end{array} \tag{21}$$

Theorem 28 *If assumptions (16), (19), (20) and (21) hold and $W, V \in C_p(\bar{Q}_0)$ are viscosity solutions of Eq. (18) in $\bar{Q}_0$ such that $\sup_{\partial^* Q}(W - V) \geq 0$, then*

$$\sup_{(t,x) \in \bar{Q}_0} e^{-\lambda t}(W(t,x) - V(t,x)) = \sup_{x \in \mathbb{R}^n} (W(T,x) - V(T,x))$$

Proof. See Theorem II.3 in [22]. □

Corollary 29 *Under the assumptions of Theorem 28, there is at most one viscosity solution V of Equation (18) satisfying the boundary condition*

$$V(T,x) = \psi(x) \quad \forall x \in \mathbb{R}^n$$

We notice that we could obtain the same result without assumptions (21) by a procedure similar to the one in Chapter V of [14] (see also Remark V.8.2 there) that also uses more modern and general tools. However this would have required to modify the proof, so we did not do so for sake of brevity and because the BSB equation satisfies also the stronger assumptions (21).

Corollary 30 *If $\psi \in C_p(\mathbb{R}^n)$, then the value function V defined in Equation (7) is the unique viscosity solution of Eq. (3) in $[0,T] \times \mathbb{R}^n_+$ with boundary conditions $V(T,\cdot) = \psi$.*

A.4 Smoothness: first derivative

From now on, since $L = 0$ in the BSB equation, we will always consider the case when $L = 0$. For results in the general case, we send the interested reader to [29].

In order to prove some smoothness of V, we have to make stronger assumptions than (20) about ψ. We assume that ψ is locally Lipschitz and that

$$|\psi_x| \leq C_4(1 + |x|^k) \tag{22}$$

for suitable constants C_4, $k \geq 0$.

The main result of this section is the following one.

Theorem 31 *If assumptions (19) and (22) holds, then $D_s V$ exists and is in $L^p_{\text{loc}}(Q_0)$ for every $p \in [1, +\infty]$ and*

$$|D_x V(t, x)| \leq M_1(1 + |x|^k)$$

for almost every $(t, x) \in Q_0$, where M_1 depends on C_1, C_2, C_3, C_4, k and T.

This result is given also in [14, Chapter V] but there only the case of C^2 final datum is considered, while we take it only locally Lipschitz and polinomially growing.

For the proof, we introduce difference quotient approximations to derivatives, that are a tool commonly used to analyse smoothness of solutions of various PDE's.

Let $u : A \to \mathbb{R}$ be a locally integrable function, where A is an open set in $[0, T] \times \mathbb{R}^n$. We call **difference quotients** of size h of u the quantities

$$D^h_\xi u(t, x) = \frac{u(t, x + h\xi) - u(t, x)}{h},$$

where ξ is a direction in $\mathbb{R}^n$, that is $|\xi| = 1$. We also indicate $D^h_i u = D^h_{e_i} u$ and $D^h_x u(t, x) = (D^h_1 u(t, x), \ldots, D^h_n u(t, x))$.

In order to prove the smoothness of V, first we will obtain bounds for $D^h_x V$, and then we will use a general result that allows us to conclude that $D_x V \in L^p_{\text{loc}}$ for $p > 1$. Let $D^h_x J = D^h_x J(t, \cdot, u)$.

Lemma 32 *There exists M_1, that depends on C_1, C_2, C_4, k and T, such that for all directions ξ,*

$$|D^h_\xi J| \leq M_1(1 + |x|^k)$$

for every $h \in (0, 1]$.

Proof. Given $(t, x_0) \in Q_0$, let X_s be the solution of (15) with $X_t = x_0$, and let X^h_s be the solution of (15) with $X_t = x_0 + h\xi$. Also, let $\Delta X_s = (X_s - X^h_s)/h$. Since ψ is Lipschitz, then it is (weakly) differentiable in each direction and the fundamental theorem of calculus holds. Then

$$D_\xi J = \mathbb{E}\left[\frac{1}{h}(\psi(X^h_T) - \psi(X_T))\right] = \mathbb{E}\left[\int_0^1 \langle \psi_x(X^\lambda_T), \Delta X_s\rangle \, d\lambda\right]$$

where $X^\lambda_s = (1 - \lambda)X_s + \lambda X^h_s$. By (22),

$$\left|\int_0^1 \psi_x(X^\lambda_T) \, d\lambda\right| \leq \int_0^1 C_4(1 + |X^\lambda_T|^k) \, d\lambda \leq M(1 + |X_T|^k + |X^h_T|^k)$$

By Cauchy-Schwartz

$$
\begin{aligned}
|D_\xi J| &= M\mathbb{E}\left[\int_t^T (1+|X_s|^k+|X_s^h|^k)^2\, ds\right]^{1/2} \mathbb{E}\left[\int_t^T |\Delta X_s|^2\, ds\right]^{1/2} + \\
&\quad + M\mathbb{E}\left[(1+|X_T|^k+|X_T^h|^k)^2\right]^{1/2} \mathbb{E}\left[|\Delta X_T|^2\right]^{1/2}
\end{aligned}
$$

We bound the first and the third term on the right hand side using Proposition 24 with $m = 2k$ and $x = x_0, x_0 + h\xi$. We have also that $\mathbb{E}[|\Delta X_s|]^2 \leq B$ (see [14] p. 400), where B depends on C_1 in (19). Since $|\xi| = 1$, $h \in (0, 1]$,

$$
1 + |x|^{2k} + |x + h\xi|^{2k} \leq c_k(1 + |x|^2 k)
$$

for suitable c_k. This implies the lemma. □

Now we are ready for the proof of the main theorem.

Proof. We take a generic open bounded set B; then by Lemma 32 we have $|D_x^h J(t,x,u)| \leq M_1(1+|x|^k)\ \forall (t,x) \in A$. Then, since these bounds are the same for all controls u, we obtain that $|D_x^h V(t,x)| \leq M_1(1+|x|^k)$. Then we take $p > 1$ and an open set A such that $B \subseteq A$ and $\mathrm{dist}(B, \partial A) < 1 \wedge T$, and we have that $D_x^h V \in L^p(B)$ and

$$
\|D_x^h V\|_{L^p(B)} \leq M_3 \|(1+|x|^k)\|_{L^p(B)}
$$

for all $h \in (0, T \wedge 1)$, where M_3 depends on M_1 and M_2. This implies (see [12], p. 246–248) that $D_x V \in L^p(B)$ and $\|D_x u\|_{L^p(B)} \leq M_3 \|(1+|x|^k)\|_{L^p(B)}$. Moreover, $D_x u$ is also the derivative in the Sobolev sense. In fact, for each $\phi \in C_0^\infty((0,T) \times \mathbb{R}^n)$,

$$
\int D_i u\phi = \int \lim_{h\to 0} D_i^h u\phi = \lim_{h\to 0} \int D_i^h u\phi = -\lim_{h\to 0}\int u D_i^h \phi = -\int u D_i \phi
$$

and the conclusion follows. □

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