Robustness of the Black-Scholes approach in the case of options on several assets

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Abstract

In this paper we analyse a stochastic volatility model that is an extension of the traditional Black-Scholes one. We price European options on several assets by using a superstrategy approach. We characterize the Markov superstrategies, and show that they are linked to a nonlinear PDE, called Black-Scholes-Barenblatt (BSB) equation. This equation is the Hamilton-Jacobi-Bellmann equation of an optimal control problem, which has a nice financial interpretation. Then we analyse the optimization problem included in the BSB equation and give some sufficient conditions in order to reduce the BSB equation to a linear Black-Scholes equation. Some examples are given.

1 Introduction

In this paper we analyse the robustness of the Black-Scholes formula in the case of European multiasset derivatives. The problem is the following; we have a riskless asset, whose value we suppose constant through time, and n risky assets whose prices $S = (S^1, \ldots, S^n)$ follow the dynamic

$$dS_t = \bar{S}_t \sigma_t \, dW_t,$$

where W is a n-dimensional Brownian motion under the so called forward-neutral measure $\mathbb{Q}$ [4], the matrix process σ takes values in a closed bounded set $\Sigma \subset M(n, n, \mathbb{R})$, and $\bar{S}_t = \text{diag}\ (S_t)$. We then consider an agent who wants to price and hedge an European contingent claim whose payoff is a continuous deterministic function $h(\cdot)$ with polynomial growth, calculated in S_T. Since the market could be incomplete because of the stochastic volatility σ and the agent is not able to hedge the volatility, he chooses to hedge the option by using the superhedging approach. Following [1] and [13], we assume that he fixes the price of the option as $C_t = C(t, S_t)$ and builds a self-financing portfolio consisting of a quantity $\Delta_t^i = \frac{\partial C}{\partial S_i}(t, S_t)$ of the asset $S^i, i = 1, \ldots, n$, where $C(t, S)$ is a solution of the following nonlinear PDE

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\max_{\gamma\in\Sigma} \operatorname{tr}\left(D^2C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*\right) = 0, & t\in[0,T), s\in\mathbb{R}^n_+ \\ \\ C(T,s) = h(s), & s\in\mathbb{R}^n_+ \end{cases} \tag{1}$$

called Black-Scholes-Barenblatt (BSB) equation in analogy with [1], where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s),\ldots,\frac{\partial C}{\partial s_n}(t,s))$, and $D^2C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (1) is an Hamilton-Jacobi-Bellmann equation, and it is linked to a stochastic control problem that have a nice financial interpretation. In fact, the agent could build a "subjective" model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \; dW_t$$

where the state variable S^γ has to be controlled by the "subjective" volatility γ in order to maximize the payoff function

$$\mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)}h(S_T^\gamma) \mid \mathcal{F}_t\right]$$

corresponding to the no-arbitrage price of the contingent claim at time t. In this way, the agent wants to maximize in γ his payoff protecting himself against "the worst possible case", and Equation (1) is the Hamilton-Jacobi-Bellmann equation corresponding to this optimal control problem. The central result of this paper is Theorem 2, where we characterize the superstrategies in terms of a quantity R_t, that could be interpreted as a consumption rate, and we prove that the price-hedge strategy above is the cheapest Markov superstrategy in a sufficiently large class. Equation (1) hides a simple but significant problem; that is, to solve the maximum problem which appears in the equation. This problem is significant because the agent has to solve it in order to implement the model in the right way. We find some necessary and sufficient conditions for the solution of the maximum problem that provide a nice geometrical interpretation of it. Moreover we find that under particular conditions it is possible to reduce the nonlinear BSB equation to a linear Black-Scholes (BS) equation. We show also that it is possible to obtain as a particular case the results of [1] and [5]. Finally we analyse some examples: Margrabe's exchange option, for which we find out that the BSB equation is always reduced to a BS equation; a European option on the geometric mean of two assets, which gives us the feeling of the behaviour of a non-convex payoff, and last, an example of a possible set Σ, suggested by a situation of stochastic correlation between two assets, where we find sufficient conditions in order to reduce the BSB equation to a BS equation.

The main papers regarding this kind of approach to stochastic volatility models are the ones of N. El Karoui, M. Jeanblanc-Piqué and S.E. Shreve (see [5]) and of M. Avellaneda, A. Lévy and A. Parás (see [1]) which deal with the superhedging problem in the case of options depending on one single asset, and the one of Lyons [13], which deals with options on several assets. In [5] the problem is widely treated in the case of European options having a payoff h convex in the asset price. For this kind of options, they succeed in giving a superstrategy depending on the solution of a particular BS equation, obtained by dominating the stochastic volatility with a deterministic function of the price. However, options are traded in the markets that have genuine non-convex payoffs, such as call-spread options, that are difference between two calls with different strike prices and same maturity. As noticed in [1] the approach in [5] is not generalizable to this kind of payoffs. To this purpose in [1] the authors introduce a nonlinear PDE, which they call Black-Scholes-Barenblatt equation, which is the 1-dimensional version of Eq. (1). In their case the set Σ is a closed interval $[\sigma_{\min}, \sigma_{\max}]$ in the real line, with $0 < \sigma_{\min} < \sigma_{\max}$. They prove a result analogous to our Theorem 2 in dimension 1. Since from many years options on several assets have a theoretical treatment (see for example Margrabe's exchange option in [14], Stulz's and Johnson's option on the maximum of two or more assets in [15] and in [9]), we treated the case of European options on several assets, as Lyons did in [13]. While Lyons uses stochastic integration without probability, in our work we use the theory of stochastic control. We present some examples of contingent claims on several assets; the first one is the well-known Margrabe's exchange option, and the second one is an option which gives a feeling of how things can work in more dimensions. For sake of simplicity, we did not include the case of interest rate derivatives as caps, floors or swaptions, which are a typical example of multiasset options, because in the case of interest rate derivatives other topics would deserve our attention. This case is anyway being studied by one of the two authors. Moreover, we choose not to study problems such as existence, uniqueness and regularity of the solution of Equation (1) in detail, but we choose to include a section presenting in a qualitative way the problem and the existing results in literature about this equation. Also this study is also being carried out by one of the two authors [8].

The paper is organized as follows: in section 2 we present the problem and give the characterization theorem; in section 3 we analyse the BSB equation and we present some results in literature; in section 4 we give sufficient conditions to solve the optimization problem, and we show how it is possible to reduce the BSB equation to a BS equation when the optimization problem has a constant solution; in section 5 we obtain as a particular case the results of [1] and [5]; in section 6 we apply our results to Margrabe's exchange option, and in section 7 we do the same to a call option on the geometric mean of two assets; in section 8 we present a possible case of set Σ, concerning a stochastic correlation problem, in which there are explicit solutions for the BSB equation.

We wish to thank Vincent Lacoste for showing us the works [5] and [1], from which we started, Dario Bini and Mauro Nacinovich for their deep understanding of the geometry of the optimization problem, and Claude Martini which suggested us to study the stochastic correlation problem. We also wish to thank an anonymous referee that pointed out many ambiguities in the first draft of this manuscript.

2 The model

We suppose that there exists a riskless asset M and n risky assets S^i, $i = 1, \ldots, n$ in the market. We make the usual assumptions that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t$ represents the information available up to time t and that M and S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Besides, we assume that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is called **forward-neutral probability** [4], such that the value of the riskless asset M remains constantly equal to 1 through time, and the dynamics of the assets S^i under $\mathbb{Q}$ are the following:

$$dS^i_t = S^i_t \langle \sigma^i_t, dW_t \rangle \tag{2}$$

where $(W_t)_t$ is a n-dimensional $\mathbb{Q}$-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^n$ and σ^i is a n-dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\Sigma)$, where Σ is a closed bounded set in the space of $n \times n$ real matrices $M(n, n, \mathbb{R})$ and $\mathcal{A}(\Sigma)$ (which we call set of **admissible volatilities**) is the set of Σ-valued processes progressively measurable with respect to $(\mathcal{F}_t)_t$.

Remark 1 We do not assume that the market is complete. In particular, the filtration $(\mathcal{F}_t)_t$ could be strictly larger than the one generated by W or than the one generated by S. In this case we have a genuine stochastic volatility model and the market is incomplete. Moreover, the assumption that the interest rate is zero can be achieved by a so-called change of numeraire, that is by expressing the prices of all the assets in the market in units of a fixed asset (in our example, in units of the riskless asset). For more details, see [4].

We can write the dynamics of the risky assets in a more compact vectorial notation in this way:

$$dS_t = \bar{S}_t \sigma_t \, dW_t$$

where:

$$\bar{S}_t = \mathrm{diag}\,(S_t) = \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S_t^n \end{pmatrix}$$

Now we consider an operator in the market that wants to create and sell a European derivative asset with payoff $h(S_T) = h(S_T^1, \dots, S_T^n)$, where h is a measurable function having linear growth. In order to price and hedge the asset, the agent fixes a price C_t for the asset at time t and builds a self-financing portfolio Π, holding η_t units of the money market account M and Δ_t^i units of the asset S^i at time t. We indicate with Π_t the value of the portfolio at time t, defined by:

$$\Pi_t = \eta_t + \langle \Delta_t, S_t \rangle \tag{3}$$

where $\Delta_t = (\Delta_t^1, \dots, \Delta_t^n)$. In order to prevent arbitrage opportunities, we say that the portfolio is **admissible** if Π is a supermartingale: in this way, if $\Pi_0 = 0$, we have $\mathbb{E}[\Pi_T] \leq 0$, so there aren't arbitrage opportunities in the market. We say that the portfolio is **self-financing** if Π follows the dynamic

$$\begin{cases} d\Pi_t = \langle \Delta_t, dS_t \rangle \\ \Pi_0 = C_0 \end{cases} \tag{4}$$

If the portfolio is self-financing, the process Δ is sufficient to characterize it, and $\eta_t = \Pi_t - \langle \Delta_t, S_t \rangle$.

The market is not complete because of the stochastic volatility σ, so the agent chooses to hedge the claim using the superhedging approach. To this aim, we define the **tracking error**:

$$e_t = \Pi_t - C_t$$

so $e_0 = 0$ by definition of Π. Intuitively, the tracking error gives the error made by the operator in estimation, or better, the difference between the hedging portfolio held by the agent and the option sold. If the portfolio is admissible and self-financing, we say that (C, Δ) is a

- **superhedging strategy**, or simply **superstrategy** if e is a non decreasing process
- **subhedging strategy**, or simply **substrategy**, if e is a non increasing process

- **hedging strategy** if e is identically equal to zero.

The sense of these definitions is the following: if one builds a superstrategy, one can successfully hedge a short position in the contingent claim C_t, while if one builds a substrategy, one can successfully hedge a long position in C_t. In particular with a superstrategy we have that $e_T \geq 0$, that is $\Pi_T \geq C_T = h(S_T)$, so the portfolio succeeds in overhedging the contingent claim. We notice that superstrategies and substrategies are good candidates to be arbitrages: in fact if we have (say) $\Pi_T \geq C_T$ $\mathbb{Q}$-a.s. and $\Pi_T > C_T$ with positive probability, then the agent succeeds in making a profit with no initial endowment. This means that the value C_t in a superstrategy has to be interpreted as an arbitrage upper bound for the price of the claim at time t; similarly, the value C_t in a substrategy has to be interpreted as a lower bound.

As already outlined by [1], [5] and [13], in order to have a superstrategy, a natural procedure is this: the agent fixes the price C_t of the option and the quantities Δ_t of the risky assets in the hedging portfolio Π at time t as

$$C_t = C(t, S_t), \qquad \Delta_t^i = \frac{\partial C}{\partial s_i}(t, S_t)$$

where C is the solution of the following partial differential equation:

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2} \max_{\gamma \in \Sigma} \operatorname{tr}\left(D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*\right) = 0, & t \in [0,T), s \in \mathbb{R}_+^n \\ C(T,s) = h(s), & s \in \mathbb{R}_+^n \end{cases} \tag{5}$$

where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s), \ldots, \frac{\partial C}{\partial s_n}(t,s))$, and $D^2C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (5), as in [1], will be called from now on the **Black-Scholes-Barenblatt equation** for h. Moreover, in order to have a substrategy it is sufficient to make as in the previous by substituting the maximum operator in Equation (5) with a minimum.

This choice has a stochastic control interpretation: we assume that the agent doesn't know the volatility σ, but in order to hedge the asset he can use the model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \, dW_t \tag{6}$$

where the process $\gamma \in \mathcal{A}(\Sigma)$ is another admissible volatility. We can interpret the γ as a control, corresponding to the subjective volatility of the agent, that he uses in order to decide his strategy. If the market was complete and the dynamics was given by Eq. (6), then the arbitrage free price of the European contingent claim would be

$$C_t^{\mathbb{Q}',\gamma} = \mathbb{E}_{\mathbb{Q}'}\left[h(S_T^\gamma) \mid S_t^\gamma\right] \tag{7}$$

Since the market is not complete, we are not sure that the contingent claim defined by h is attainable. In particular, we have two sources of uncertainity, that are the particular forward-neutral measure $\mathbb{Q}$ and the volatility γ. Since the agent doesn't know either of these objects, he associates the price of the claim with the interval

$$\left\{ \mathbb{E}_{\mathbb{Q}'}\left[e^{-r(T-t)} h(S_T^\gamma) \middle| \mathcal{F}_t \right] \;\middle|\; \mathbb{Q}' \simeq \mathbb{P} \text{ such that Eq. (6) holds under } \mathbb{Q}' \text{ and } \gamma \in \mathcal{A}(\Sigma) \right\}$$

in the real line, called the **set of admissible prices**. However, since $\gamma \in \mathcal{A}(\Sigma)$, for all the forward-neutral measures $\mathbb{Q}$ and $\gamma \in \mathcal{A}(\Sigma)$ we have

$$V_{\mathbb{Q}}^-(t, S_t^\gamma) \leq C_t^{\mathbb{Q},\gamma} \leq V_{\mathbb{Q}}^+(t, S_t^\gamma)$$

where

$$V_{\mathbb{Q}}^{-}(t,s) = \inf_{\gamma\in\mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma)\middle|\, S_t^\gamma = s\right] \tag{8}$$

and

$$V_{\mathbb{Q}}^{+}(t,s) = \sup_{\gamma\in\mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma)\middle|\, S_t^\gamma = s\right] \tag{9}$$

are the value functions of two optimal control problems, having as payoff function the right-hand side of Eq. (7). In particular, under the assumptions we made, $V_{\mathbb{Q}}^{+}$ is the unique viscosity solution of Equation (5) (see Section 3) and doesn't depend on the particular choice of the measure $\mathbb{Q}$ (so we can drop the subscript in $\mathbb{Q}$). So we have that $[V^-(t,S_t), V^+(t,S_t)]$ defines the interval of admissible arbitrage prices for the claim at time t, in the sense that if the price of the claim lies outside this interval, then it is possible to build an arbitrage in the market. This can be interpreted as follows: when the agent decides the price of the claim, he wishes to protect himself against the worst possible outcome, and he chooses the price that corresponds to this. Moreover, we can interpret the γ as a control, corresponding to the subjective volatility of the agent, that he uses in order to decide his strategy.

We first show a superstrategy. We indicate with $C_p^{1,2}((0,T)\times\mathbb{R}_+^n)$ the space of functions C that are continuous and with polynomial growth on $(0,T)\times\mathbb{R}_+^n$ together with their first derivative in t and first and second derivatives in s.

Theorem 2 *If we restrict ourselves to the case $C_t = C(t,S_t)$, with $C \in C_p^{1,2}([0,T)\times\mathbb{R}_+^n)$, then:*

i) (C,Δ) is a superstrategy if and only if $\Delta_t = DC(t,S_t)$ and

$$R_t = -\frac{\partial C}{\partial t}(t,S_t) - \frac{1}{2}\mathrm{tr}\ D^2C(t,S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*$$

is non negative $\mathbb{Q}$-a.s. for all $t \in [0,T]$.

ii) If C is the solution of Equation (5) and $\Delta_t = DC(t,S_t)$, then (C,Δ) is a superstrategy. Moreover there does not exist another superstrategy (C',Δ') such that $C'_t < C(t,S_t^\gamma)$ with positive probability for some t and for all $\gamma\in\mathcal{A}(\Sigma)$.

Proof. The evolution of $(e_t)_t$ is the following:

$$\begin{aligned} de_t &= \langle\Delta_t, dS_t\rangle - \frac{\partial C}{\partial t}(t,S_t)\ dt - \langle DC(t,S_t), dS_t\rangle - \frac{1}{2}\mathrm{tr}\ \left(D^2C(t,S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*\right)\ dt = \\ &= R_t\ dt + \langle\Delta_t - DC(t,S_t), dS_t\rangle \end{aligned}$$

Now, in order for the process e to be increasing, it must have finite variation, so we must have $\Delta_t = DC(t,S_t)$; besides this condition, e is increasing if and only if $R_t \geq 0$ $\mathbb{Q}$-a.s. $\forall t$. Conversely, if these two conditions are satisfied, then e is increasing. At last, Δ is admissible. In fact, since $C\in C_p^{1,2}$, we have that

$$\mathbb{E}\left[\int_0^T \|\sigma_t^*\bar{S}_t\Delta_t\|^2\ dt\right] \leq M\mathbb{E}\left[\int_0^T \|S_t\|^2C^2(1+\|S_t\|^m)^2\ dt\right]$$

for some C and m, where $M = \sup\{\lambda \mid \|\gamma^*x\| \leq \lambda\|x\| \quad \forall x\in\mathbb{R}^n, \gamma\in\Sigma\}$. Since for all $m\geq 2$ we have $\mathbb{E}[\|S_t\|^m] \leq B_m(1+\|S_0\|^m)$ for a suitable B_m (see [10]), we have $\mathbb{E}[\int_0^T \|\sigma_t^*\bar{S}_t\Delta_t\|^2\ dt] < +\infty$, so $\Pi_t = \int_0^t\langle\Delta_u, dS_u\rangle = \int_0^t\langle\sigma_u^*\bar{S}_u\Delta_u, dW_u\rangle$ is a martingale.

In order to prove (ii), we notice that under its assumptions, we have:

$$R_t = \frac{1}{2}\left(\max_{\gamma\in\Sigma} \mathrm{tr}\ \left(D^2C(t,S_t)(\bar{S}_t\gamma)(\bar{S}_t\gamma)^*\right) - \mathrm{tr}\ \left(D^2C(t,S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*\right)\right) \geq 0$$

Besides, C is a classical solution of Eq. (5), so it is also a viscosity solution. Since the unique viscosity solution of Eq. (5) is the value function (9), then C is equal to the value function. Now we assume that there exists another superstrategy (C', Δ') such that $C_0 = \Pi_0 < C(0, S_0)$. We put $\varepsilon = C(0, S_0) - C_0$; then Π is a supermartingale, so $\Pi_0 \geq \mathbb{E}[\Pi_T]$. On the other hand, we also have that

$$C(0, S_0) = \max_{\gamma\in\Sigma} \mathbb{E}[h(S_t^\gamma)]$$

We take $\bar{\gamma}$ such that $\mathbb{E}[h(S_t^{\bar{\gamma}})] > C(0, S_0) - \varepsilon = \Pi_0$; then $\mathbb{E}[h(S_t^{\bar{\gamma}})] > \mathbb{E}[\Pi_T]$. This means that $\mathbb{P}\{h(S_T^{\bar{\gamma}}) > \Pi_T\} > 0$, and we have the result. $\diamond$

We just obtained a characterization theorem of the superstrategies. In fact, provided the value function $C \in C_p^{1,2}$, the theorem gives the minimal superstrategy, in the sense that there doesn't exist a cheaper strategy that succeeds in giving an increasing tracking error for all $\gamma \in \mathcal{A}(\Sigma)$. Since the agent doesn't know the true volatility σ, he must protect himself against all the possible volatilities $\gamma \in \mathcal{A}(\Sigma)$ and this is the best result he can expect.

The quantity R_t can be seen as a consumption instantaneous rate obtained by the superstrategy of the agent. A discussion on the financial interpretation of R can be found in [13].

Analogously, we can analyse the problem of finding a substrategy.

Theorem 3 *If we restrict ourselves to the case $C_t = C(t, S_t)$, with $C \in C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$, then:*

i) *(C, Δ) is a substrategy if and only if $\Delta_t = DC(t, S_t)$ and*

$$R_t = -\frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2}\mathrm{tr}\ D^2C(t,S_t)(\bar{S}_t\sigma_t)(\bar{S}_t\sigma_t)^*$$

is non positive $\mathbb{Q}$-a.s. for all $t \in [0, T]$.

ii) *If C is the solution of Equation (5) (provided we substitute the* max *with* min*), and $\Delta_t = DC(t, S_t)$, then (C, Δ) is a substrategy. Moreover there does not exist another superstrategy (C', Δ') such that $C'_t < C(t, S_t^\gamma)$ with positive probability for some t and for all $\gamma \in \mathcal{A}(\Sigma)$.*

Proof. The proof is similar to that of Theorem 2. $\diamond$

We notice that the substrategy case is in some sense symmetric to the superstrategy one. This has also a simple mathematical reason: in fact, if C is the solution of the BSB equation with final condition h, then $-C$ is the solution of the BSB equation with min instead of max, and final condition $-h$. In other words, if C defines a superstrategy for the claim defined by h, then $-C$ defines a substrategy for the claim defined by $-h$. This allows us to analyse only one of the two cases, and to have in an automatic way results for the other one. For this reason, in the rest of the paper we will analyse only the superstrategy case, and it will be implicit that analogous results will hold also for the substrategy.

3 The BSB equation

We have seen that in order to have a Markov superstrategy, we need a $C^{1,2}$ solution of Eq. (5). To this purpose, we make a brief survey of the results existing in literature about nonlinear parabolic equations. We don't want to make rigorous proofs of our assertions, but only to give some ideas of how things can be done. A rigorous treatment of the problem is the topic of an ongoing work [8] by one of the two authors.

There are two main approaches to study this kind of equation. The first one is by classical methods, and the main reference is Krylov's book [11]. The problem is that with this approach it is required that $h \in C_b^3(\mathbb{R}^n)$ (see for example [7] p. 169 and [11] p. 301), while the typical h characterizing a European asset is C^0 and with linear growth. Moreover, this theory is quite difficult to handle for non-specialists.

Much more can be done with the other dominant approach in dealing with nonlinear PDEs, that is the notion of viscosity solutions. This theory, developed by Crandall and Lions in a series of papers in the early '80s, allows for powerful existence and uniqueness theorems, with proofs that are much easier than the ones of classical theorems. Moreover, there are interesting links between optimal control problems and viscosity solutions of the corresponding HJB equations: in particular, under technical assumptions, the value function is the unique viscosity solution of the HJB equation. The interested reader can see [3] for a detailed introduction to the concept of viscosity solutions and its applications, and [7] for the links between optimal stochastic control and viscosity solutions of HJB equations. However, viscosity solutions are a class of generalized solutions to nonlinear PDEs that are only continuous a priori; so if we want to apply Theorem 2 with a viscosity solution, we also need to prove that the solution is smooth enough to apply Itô formula.

The first step in using the theory of viscosity solutions is an existence and uniqueness theorem.

Theorem 4 *If h has polynomial growth, then the value function C defined by Equation (9) is a viscosity solution of Eq. (5) in $[0,T] \times \mathbb{R}_+^n$. Moreover, it is the unique viscosity solution having polynomial growth that satisfy the boundary condition $C(T,s) = h(s)$ for all $s \in \mathbb{R}_+^n$.*

Proof. The proof follows the standard technique in the theory of viscosity solutions. For the existence, it is sufficient to prove that the value function (9) is continuous and that the dynamic programming principle holds for the control problem. The uniqueness is based on a standard comparison result. $\diamond$

The second step is to prove smoothness of the viscosity solution. The main reference in this topic is Wang's work [16]. In order to use his results, we have to make a change of variable. We put $y_i = \log s_i$. Then the BSB equation becomes:

$$\begin{cases} \dfrac{\partial \tilde{C}}{\partial t}(t,y) + \dfrac{1}{2} \max_{\gamma \in \Sigma} \operatorname{tr}((D^2 \tilde{C}(t,y) - \operatorname{diag}(D\tilde{C}(t,y)))\gamma\gamma^*) = 0, & t \in [0,T), y \in \mathbb{R}^n \\ \tilde{C}(T,y) = \tilde{h}(y), & y \in \mathbb{R}^n \end{cases} \tag{10}$$

where $\underline{1} = (1,\ldots,1)$ and $\tilde{C}(t,y) = C(t,(e^{y_i})_i) = C(t,s)$. In order to use Wang's results, we have to give this definition. We say that Equation (10) is **uniformly parabolic** if there exist real numbers $M > m > 0$ such that forall $\gamma \in \Sigma$ and $\xi \in \mathbb{R}^n$,

$$m\|\xi\|^2 \le \langle \gamma\gamma^*\xi, \xi \rangle \le M\|\xi\|^2$$

Since $\langle \gamma\gamma^*\xi, \xi\rangle = \langle \gamma^*\xi, \gamma^*\xi\rangle = \|\gamma^*\xi\|^2$, the definition can be rewritten in this way:

$$\exists M' > m' > 0 \text{ such that } \forall \gamma \in \Sigma, \xi \in \mathbb{R}^n, m'\|\xi\| \leq \|\gamma^*\xi\| \leq M'\|\xi\|$$

This implies that if there exists a $\gamma \in \Sigma$ which is not invertible, then Eq. (6) is not uniformly parabolic. In fact if we take $\xi \in \ker \gamma^*$, then $\|\gamma^*\xi\| = 0$. We show that also the converse is true.

Lemma 5 *If Σ is closed and bounded in $M(n, n, \mathbb{R})$, then Eq. (10) is uniformly parabolic if and only if $\Sigma \subseteq \mathrm{GL}(n, \mathbb{R})$, where*

$$\mathrm{GL}(n, \mathbb{R}) = \{\gamma \in M(n, n, \mathbb{R}) \mid \det \gamma \neq 0\}$$

Proof. As seen before, if $\exists \gamma \in \Sigma$ such that $\gamma \notin \mathrm{GL}(n, \mathbb{R})$, then Eq. (10) is not uniformly parabolic. Conversely, if $\Sigma \subseteq M(n, n, \mathbb{R})$, then $\gamma\gamma^*$ is positive definite for all $\gamma \in \Sigma$; besides, the function that goes from γ to the least eigenvalue of $\gamma\gamma^*$ is continuous, so it has a minimum $m' > 0$ in the compact set Σ; also the function going from γ to the greater eigenvalue of $\gamma\gamma^*$ is continuous, so it has a maximum $M' > 0$ in Σ, so the result follows. $\diamond$

This allows us to state the following theorem.

Theorem 6 *If Equation (10) is uniformly parabolic, and we call $\tilde{C}$ its viscosity solution, then there exists $\alpha \in (0, 1]$ such that $\tilde{C} \in C^{1,2,\alpha}(\mathbb{R}^n)$.*

This is a particular case of a general result in [16], and means that $\tilde{C}$ has second derivatives in s that are α-Hölder continuous. This means that also $C(t, s_1, \ldots, s_n) = \tilde{C}(t, \log s_1, \ldots, \log s_n)$ is C^2, so we can apply Itô formula. The final conclusion is that if the set of volatilities Σ is a closed bounded set in $\mathrm{GL}(n, \mathbb{R})$, then there exists a Markov superstrategy, and it is given by the solution of Eq. (5).

Remark 7 Uniform parabolicity of Eq. (10) is needed to have regularity of solutions; indeed, there are significant cases (for example, the one treated in section 7) in which Eq. (10) is not uniformly parabolic, but it still has $C^{1,2}$ solutions. A work about this topic [8] has been carried out by one of the authors.

4 The optimization problem

We notice that Equation (5) contains an optimization problem. The agent has to solve this problem if he wants to solve numerically the BSB equation, and if he wants to have a view of which could be the "worst case" against him. Moreover, it turns out that in some cases the problem gives a simple way to pass from the BSB equation to a Black-Scholes equation, which is simpler both from a theoretical point of view as from the numerical side. Anyway, this problem is significant only if $C \in C^{1,2}$, so in the following we will assume it.

The problem can be seen as an optimization problem of a bilinear form in the spaces of $n \times n$ real matrices, so we can characterize the solution and if Σ is smooth enough we can solve the problem explicitly.

The optimization problem is

$$\max_{\gamma \in \Sigma} F_{t,s}(\gamma) \tag{11}$$

where $F_{t,s} : \Sigma \to \mathbb{R}$ is defined by

$$F_{t,s}(\gamma) = \mathrm{tr}\ \left(\bar{s}D^2C(t,s)\bar{s}\gamma\gamma^*\right) = \mathrm{tr}\ \left(A_{t,s}\gamma\gamma^*\right) \tag{12}$$

where $A_{t,s} = \bar{s}D^2C(t,s)\bar{s}$ and C is the solution of Eq. (5). We will often write F and A instead of $F_{t,s}$ and $A_{t,s}$ if there is no risk of ambiguity. This is an optimization problem of a bilinear form in the closed bounded set $\Sigma \subset M(n,n,\mathbb{R})$. This allows us to make a first statement.

Lemma 8 *If $0 \notin \Sigma$, then F assumes maximum on $\partial\Sigma$.*

Proof. Since F is continuous and Σ is a compact set, then F has a maximum $\bar{\gamma}$ in Σ. Let's suppose that $F(\bar{\gamma}) > 0$ and $\bar{\gamma}$ is in $\overset{\circ}{\Sigma}$. Then there exists an $\varepsilon > 0$ such that $(1+\varepsilon)\bar{\gamma} \in \Sigma$, and

$$F((1+\varepsilon)\bar{\gamma}) = (1+\varepsilon)^2 F(\bar{\gamma}) > F(\bar{\gamma})$$

so $\bar{\gamma}$ is not maximum, which is a contradiction. In an analogous way, if $F(\bar{\gamma}) < 0$ and $\bar{\gamma}$ is in $\overset{\circ}{\Sigma}$, then we can repeat the above argument with $(1-\varepsilon)\,\bar{\gamma}$, reaching a contradiction. Finally if $F(\bar{\gamma}) = 0$ we can find a $\gamma^* \in \partial\Sigma$ such that $F(\gamma^*) = 0$, so we have the result. $\diamond$

Luckily, it would be unrealistic to include 0 in Σ, because if $0 \in \Sigma$, then there is the possibility of a null volatility in all the assets. If $0 \notin \Sigma$, we can reformulate the problem in this way:

$$\max_{\gamma \in \partial\Sigma} F(\gamma) \tag{13}$$

Now we present a first-order condition that has to be satisfied if the set Σ has smooth boundary.

Proposition 9 *If $\partial\Sigma$ is smooth (C^1) and it is defined by $G(\gamma) = 0$ and $\bar{\gamma}$ is a maximum for F, then $\exists \lambda \in \mathbb{R}$ such that $2A\bar{\gamma} - \lambda DG(\bar{\gamma}) = 0$, where DG indicates the gradient with respect to γ.*

Proof. The result follows immediately from an application of the Lagrange multiplier method and from the fact that $DF(\gamma) = 2A\gamma$. In fact

$$F(\gamma) = \mathrm{tr}\ (A\gamma\gamma^*) = \mathrm{tr}\ \left(\left(\sum_{j,k=1}^{n} a_{lk}\gamma_{ij}\gamma_{kj}\right)_{il}\right) = \sum_{i,j,k=1}^{n} a_{ik}\gamma_{ij}\gamma_{kj}$$

so the derivative of F with respect to γ_{pq} is

$$\partial_{pq}F(\gamma) = \sum_{k=1}^{n} a_{kp}\gamma_{kq} + \sum_{i=1}^{n} a_{pi}\gamma_{iq} = 2\sum_{k=1}^{n} a_{kp}\gamma_{kq} = 2\langle a^p, \gamma^q\rangle$$

where a^p is the p-th row of A and γ^q is the q-th row of γ; this gives the result. $\diamond$

Remark 10 We have seen that if $\partial\Sigma$ is defined by $G(\gamma) = 0$, with $G \in C^1(M(n,n,\mathbb{R}),\mathbb{R})$, then $\bar{\gamma}$ is a solution of the system:

$$\begin{cases} 2A\gamma - \lambda DG(\gamma) = 0 \\ G(\gamma) = 0 \end{cases}$$

In particular, $\bar{\gamma}$ depends continuously on A. Viceversa, if $\partial\Sigma$ is not regular, the dependence of $\bar{\gamma}$ on A may not necessary be continuous.

An important case is when the optimization problem has a constant solution. In this case, we can reduce the non linear BSB equation to a standard Black-Scholes equation.

Proposition 11 *If $C \in C^{1,2}([0,T) \times \mathbb{R}^n)$ is a solution of the Black-Scholes equation*

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\text{tr}\,(D^2C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*) = 0, & t \in [0,T), s \in \mathbb{R}^n_+ \\ C(T,s) = h(s), & s \in \mathbb{R}^n_+ \end{cases} \tag{14}$$

with $\bar{\gamma} \in \Sigma$, such that the problem (11) attains its maximum in $\bar{\gamma}$ for all $(t,s) \in [0,T) \times \mathbb{R}^n_+$, then C is also a solution of Equation (5).

Proof. The maximum problem (11) has solution

$$\text{tr}\,(D^2C(t,s)(s\bar{\gamma})(s\bar{\gamma})^*) = \max_{\gamma\in\Sigma} \text{tr}\,(D^2C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*)$$

so the conclusion follows. ◇

This proposition is very useful to reduce the BSB equation, for which explicit solutions are very rare, to a BS equation, for which explicit solutions are very common; we will see an example of this case in the next sessions.

5 The one-dimensional case

Now we analyse with more detail the case in which $n = 1$. This was the first case studied in literature and the main references are [1] and [5]. The first thing we notice is that the typical example of Σ in this case is $\Sigma = [\sigma_{\min}, \sigma_{\max}]$, with $0 < \sigma_{\min} < \sigma_{\max}$. For this reason, throughout all this section we will assume that $n = 1$ and $\Sigma = [\sigma_{\min}, \sigma_{\max}]$.

We obtain the following results, which are already present in [1].

Lemma 12 *The problem (11) has solution*

$$\sigma^+\left(\frac{\partial^2 C}{\partial s^2}(t,s)\right) = \begin{cases} \sigma_{\max} & \text{if } \dfrac{\partial^2 C}{\partial s^2}(t,s) \geq 0 \\ \sigma_{\min} & \text{if } \dfrac{\partial^2 C}{\partial s^2}(t,s) < 0 \end{cases}$$

Proof. In this case, $F_{t,s}(\gamma) = s^2\frac{\partial^2 C}{\partial s^2}(t,s)\gamma^2$, so the conclusion follows. ◇

Corollary 13 *The BSB equation becomes*

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\sigma^+\left(\dfrac{\partial^2 C}{\partial s^2}(t,s)\right)^2 s^2\dfrac{\partial^2 C}{\partial s^2}(t,s) = 0, & t \in [0,T), s \in \mathbb{R}^n_+ \\ C(T,s) = h(s), & s \in \mathbb{R}^n_+ \end{cases}$$

Following [5], we cite these powerful results concerning the case in which h is convex.

Lemma 14 *If the solution C is convex in s, then the problem (11) has solution $\sigma_{\max}$, so the BSB equation (5) reduces to the BS equation (14).*

This lemma is very useful thanks to the

Theorem 15 (El Karoui, Jeanblanc-Piqué, Shreve) *If $n = 1$ and h is convex, then the solution C of the BS equation (14) is convex in s.*

Corollary 16 *If h is convex and the hypotheses above hold, then the solution to the BS equation (14) with volatility $\sigma_{\max}$ is also solution of the BSB equation (5), thus giving a superstrategy.*

Proof. It follows from Theorem 15, Lemma 14 and Proposition 11. $\diamond$

6 Margrabe's exchange option

We consider Margrabe's exchange option (see [14]), whose final payoff is the following:

$$h(s) = (s_1 - \lambda s_2)^+$$

We try to apply the results of Section 4 to this option. To this purpose, for a generic $\gamma \in \Sigma$ we calculate the Black-Scholes price $C(t, s_1, s_2)$ of the exchange option, which is the solution of Eq. (14) when we substitute $\bar{\gamma}$ with

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \tag{15}$$

By Proposition 11, if the Black-Scholes price is such that the optimization problem (11) attains its maximum at γ for all $t \in [0, T)$, $s \in \mathbb{R}^2_+$, then $C(t, s_1, s_2)$ is also a solution of the BSB equation (5).

The Black-Scholes price of this option is:

$$C(t, s_1, s_2) = s_1 N(d_1) - \lambda s_2 N(d_2) \tag{16}$$

where

$$d_1 = \frac{1}{\Gamma\sqrt{T-t}} \ln\left(\frac{s_1}{\lambda s_2}\right) + \frac{1}{2}\Gamma\sqrt{T-t}, \quad d_2 = d_1 - \Gamma\sqrt{T-t}$$

$$\Gamma = \sqrt{(\gamma_{11} - \gamma_{21})^2 + (\gamma_{12} - \gamma_{22})^2}$$

and N is the cumulative distribution function of a centered reduced Gaussian random variable:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}}\, dx$$

(for a proof, see [14]). In order to apply Proposition 11, we calculate the matrix $A_{t,s}$. The first derivatives are:

$$\frac{\partial C}{\partial s_1} = N(d_1), \quad \frac{\partial C}{\partial s_2} = -\lambda N(d_2),$$

The second derivatives are the following:

$$\frac{\partial^2 C}{\partial s_1^2} = \frac{\partial d_1}{\partial s_1} N'(d_1) = \frac{1}{s_1 \Gamma \sqrt{T-t}} N'(d_1)$$

$$\frac{\partial^2 C}{\partial s_2^2} = \lambda \frac{\partial d_2}{\partial s_2} N'(d_2) = \lambda \frac{1}{s_2 \Gamma \sqrt{T-t}} N'(d_2)$$

because

$$\frac{\partial d_i}{\partial s_1} = \frac{1}{s_1 \Gamma \sqrt{T-t}}, \quad \frac{\partial d_i}{\partial s_2} = -\frac{1}{s_2 \Gamma \sqrt{T-t}}, \quad i = 1, 2$$

We have two ways to calculate the cross derivative:

$$\frac{\partial^2 C}{\partial s_1 \partial s_2} = \frac{\partial d_1}{\partial s_2} N'(d_1) = -\lambda \frac{\partial d_2}{\partial s_1} N'(d_2) = -\frac{N'(d_1)}{s_2 \Gamma \sqrt{T-t}} = -\frac{\lambda N'(d_1)}{s_2 \Gamma \sqrt{T-t}}$$

Since $C \in C^{1,2}$, from this we get that $s_1 N'(d_1) = \lambda s_2 N'(d_2)$; then we have that:

$$A_{t,s} = \bar{s} \cdot D^2 C(t,s) \cdot \bar{s} = \begin{pmatrix} s_1^2 \dfrac{\partial^2 C}{\partial s_1^2} & s_1 s_2 \dfrac{\partial^2 C}{\partial s_1 s_2} \\ s_1 s_2 \dfrac{\partial^2 C}{\partial s_1 s_2} & s_2^2 \dfrac{\partial^2 C}{\partial s_2^2} \end{pmatrix} = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

We notice that $A_{t,s}$ is proportional $\forall (t,s) \in [0,T) \times \mathbb{R}_+^2$ to the constant matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

This means that the solution to the problem (11) is constant. Hence we obtain easily the following proposition.

Proposition 17 *If $\bar{\gamma}$ is the maximum of $\|\gamma_1 - \gamma_2\|_{\mathbb{R}^2}$ on Σ, where γ_1, γ_2 are the rows of γ, then (16) is solution of Equation (5).*

Proof. We have that $\forall a, b, c \in \mathbb{R}$:

$$\operatorname{tr}\left(A_{t,s} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} (a - 2b + c)$$

so, by easy calculations, we obtain that

$$\operatorname{tr}\ (A_{t,s} \gamma \gamma^*) = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} \|\gamma_1 - \gamma_2\|_{\mathbb{R}^2}^2$$

The result follows then from Proposition 11. $\diamond$

7 An example of a non-convex payoff: option on the geometric mean

We consider an option whose final payoff is the following:

$$h(s) = (\sqrt{s_1 s_2} - K)^+$$

As in the previous section, we try to apply the results of section 4 to this option. To this purpose, we first calculate the Black-Scholes price in the case in which the dynamics is given by

$$dS_t = rS_t \ dt + \bar{S}_t \gamma \ dW_t \tag{17}$$

where γ is deterministic and given by Eq. (15).

The dynamics of $\sqrt{S_i}$ is:

$$d\sqrt{S_i(t)} = \frac{1}{2}\sqrt{S_i(t)}\gamma_1 \ dW(t) - \frac{1}{8}\sqrt{S_i(t)}\|\gamma_1\|^2 \ dt$$

The dynamics of $\sqrt{S_1 S_2}$ is:

$$d\sqrt{S_1(t)S_2(t)} = \sqrt{S_1(t)S_2(t)}\left(\frac{1}{2}\langle\gamma_1 + \gamma_2, dW(t)\rangle + \mu \ dt\right)$$

where $\mu = -\frac{1}{8}\|\gamma_1 - \gamma_2\|^2$. We can interpret $\sqrt{S_1 S_2}$ as an asset having instantaneous rate of return equal to μ, so we can apply the Black-Scholes formula and obtain the price of the option as

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\left(\sqrt{S_1(T)S_2(T)} - K\right)^+\right] &= e^{\mu(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{-\mu(T-t)}\left(\sqrt{S_1(T)S_2(T)} - K\right)^+\right] = \\
&= e^{\mu(T-t)}\left(\sqrt{S_1(t)S_2(t)}N(d_1) - Ke^{-\mu(T-t)}N(d_2)\right)
\end{aligned}$$

where

$$d_1 = \frac{1}{\tilde{\gamma}_t}\ln\frac{\sqrt{S_1(t)S_2(t)}}{Ke^{-\mu(T-t)}} + \frac{1}{2}\tilde{\gamma}_t, \quad d_2 = d_1 - \tilde{\gamma}_t, \quad \tilde{\gamma}_t = \frac{1}{2}\|\gamma_1 + \gamma_2\|\sqrt{T-t}$$

The price of the option is then:

$$C(t, s_1, s_2) = A\left(\sqrt{s_1 s_2}N(d_1) - Ke^{-\mu(T-t)}N(d_2)\right) \tag{18}$$

where $A = e^{-\frac{1}{4}\|\gamma_1 - \gamma_2\|^2(T-t)}$. Now we calculate $A_{t,s}$. The first derivatives of C are

$$\frac{\partial C}{\partial s_1} = \frac{A\sqrt{s_2}}{2\sqrt{s_1}}N(d_1), \quad \frac{\partial C}{\partial s_2} = \frac{A\sqrt{s_1}}{2\sqrt{s_2}}N(d_1)$$

The second derivatives of the price are:

$$\frac{\partial^2 C}{\partial s_1^2} = \frac{A\sqrt{s_2}}{4s_1^{3/2}}\left(\frac{N'(d_1)}{\tilde{\gamma}_t} - N(d_1)\right), \quad \frac{\partial^2 C}{\partial s_2^2} = \frac{A\sqrt{s_1}}{4s_2^{3/2}}\left(\frac{N'(d_1)}{\tilde{\gamma}_t} - N(d_1)\right)$$

$$\frac{\partial^2 C}{\partial s_1 \partial s_2} = \frac{A}{4\sqrt{s_1 s_2}}\left(\frac{N(d_1)}{\tilde{\gamma}_t} + N(d_1)\right)$$

because

$$\frac{\partial d_1}{\partial s_i} = \frac{1}{2\tilde{\gamma}_t s_i}, \quad i = 1, 2$$

We have that:

$$\begin{aligned} A_{t,s} = \bar{s} \cdot D^2 C(t,s) \cdot \bar{s} &= \frac{A\sqrt{s_1 s_2}}{4} \begin{pmatrix} \frac{N'(d_1)}{\tilde{\gamma}_t} - N(d_1) & \frac{N'(d_1)}{\tilde{\gamma}_t} + N(d_1) \\ \frac{N'(d_1)}{\tilde{\gamma}_t} + N(d_1) & \frac{N'(d_1)}{\tilde{\gamma}_t} - N(d_1) \end{pmatrix} = \\ &= \frac{\sqrt{s_1 s_2}}{4} \left(\frac{N'(d_1)}{\tilde{\gamma}_t} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - N(d_1) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right) \end{aligned}$$

We notice that this time the linear form is proportional to the linear combination of two constant matrices. In particular, we have that

$$A = \begin{pmatrix} \alpha - \beta & \alpha + \beta \\ \alpha + \beta & \alpha - \beta \end{pmatrix}$$

where

$$\alpha = \frac{\sqrt{s_1 s_2}}{4} \frac{N'(d_1)}{\tilde{\gamma}_t} > 0, \quad \beta = \frac{\sqrt{s_1 s_2}}{4} N(d_1) > 0$$

If $\partial\Sigma$ is regular, then there are few possibilities that the solution $\bar{\gamma}$ of the maximum problem 11 is constant, because it depends continuously from A, so from α and β, that is on s_1, s_2 and t; but if $\partial\Sigma$ is not regular, it can happen that things change, as we will see in the next session.

8 Stochastic correlation

Now we assume that

$$\Sigma = \left\{ \gamma \in \mathbb{R}^{2\times 2} \;\middle|\; \gamma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}, \rho \in [-1, 1] \right\}$$

where $\sigma_1 > \sigma_2 > 0$. This corresponds to the case when the volatilities σ_1, σ_2 of the two assets S_1 and S_2 are known, but the correlation is not. In particular we allow S_1 and S_2 to be in all the states between perfectly positively correlated ($\rho = 1$) and perfectly negatively correlated ($\rho = -1$), passing through a null correlation ($\rho = 0$). We indicate with S_1 the asset having higher volatility σ_1. Then we have that

$$\gamma\gamma^* = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \left(1-\rho^2\right)\sigma_2^2 \end{pmatrix}$$

We represent $A_{t,s} = A$ as

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Lemma 18 *If $|b\sigma_1| > |c\sigma_2|$, then:*

- *if $b > 0$, then the optimum is $\bar{\rho} = 1$*
- *if $b < 0$, then the optimum is $\bar{\rho} = -1$*

Proof. The function to maximize becomes:

$$f(\rho) = \mathrm{tr}\,(A\gamma\gamma^*) = a\sigma_1^2 + c\sigma_2^2 + 2b\rho\sigma_1\sigma_2 - c\sigma_2^2\rho^2$$

We have that, if $c \neq 0$, then f is a parabola, so

$$f'(\rho) = 2b\sigma_1\sigma_2 - 2c\sigma_2^2\rho$$

The vertex of f is $\rho^* = \frac{b\sigma_1}{c\sigma_2}$; if $|\frac{b\sigma_1}{c\sigma_2}| > 1$, it means that $\rho^* \notin [-1,1]$, so the maximum is in the boundary of $[-1,1]$, that is $\bar{\rho} \in \{-1,1\}$. We suppose that $b > 0$; if $c > 0$, then $\rho^* > 1$, is the maximum of the parabola, so f is increasing in $[-1,1]$ and $\bar{\rho} = 1$; if $c < 0$, then $\rho^* < -1$ is the minimum of the parabola, so f is increasing in $[-1,1]$ and again $\bar{\rho} = 1$; at last if $c = 0$, f is a line with slope $2b\sigma_1\sigma_2 > 0$, so $\bar{\rho} = 1$. The case $b < 0$ is obtained with similar arguments. $\diamond$

Corollary 19 (Margrabe's exchange option) *If $C(t, s_1, s_2)$ is given by Equation (16), then the optimal $\bar{\rho}$ is $\bar{\rho} = -1$, and the BSB equation becomes a BS equation with volatility*

$$\begin{pmatrix} \sigma_1 & 0 \\ -\sigma_2 & 0 \end{pmatrix}$$

Proof. We have $\frac{b\sigma_1}{c\sigma_2} = -\frac{\sigma_1}{\sigma_2} < -1$ and $b < 0$, then we apply Lemma 18 and Proposition 11. $\diamond$

Corollary 20 (option on the geometrical mean) *If $C(t, s_1, s_2)$ is given by Equation (18), then the optimal $\bar{\rho}$ is $\bar{\rho} = 1$, and the BSB equation becomes a BS equation with volatility*

$$\begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & 0 \end{pmatrix}$$

Proof. We have $b = \alpha + \beta$ and $c = \alpha - \beta$. Since $\alpha, \beta > 0$, then $|\alpha + \beta| > |\alpha - \beta|$ and $\frac{\sigma_1}{\sigma_2} > 1$. Since $b = \alpha + \beta$, we may apply Lemma 18 and Proposition 11. $\diamond$

In both cases we have seen that to apply Lemma 18 is necessary that $\sigma_1 > \sigma_2$. For this it is sufficient to indicate with S_1 the asset having higher volatility.

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