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Robustness of the Black-Scholes approach in the case of options on several assets*

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Abstract. In this paper we analyse a stochastic volatility model that is an extension of the traditional Black-Scholes one. We price European options on several assets by using a superstrategy approach. We characterize the Markov superstrategies, and show that they are linked to a nonlinear PDE, called the Black-Scholes-Barenblatt (BSB) equation. This equation is the Hamilton-Jacobi-Bellman equation of an optimal control problem, which has a nice financial interpretation. Then we analyse the optimization problem included in the BSB equation and give some sufficient conditions for reduction of the BSB equation to a linear Black-Scholes equation. Some examples are given.

Key words: stochastic volatility, superreplication, stochastic optimal control, Hamilton-Jacobi-Bellman equations

JEL classification: G13

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1. Introduction

In this paper we analyse the robustness of the Black-Scholes formula with respect to stochastic volatility in the case of European multiasset derivatives. The

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problem is the following; we have a riskless asset, whose value we suppose constant through time, and n risky assets whose prices $S = (S^1, \dots, S^n)$ follow the dynamic

$$dS_t = \bar{S}_t \sigma_t \, dW_t \ ,$$

where W is an n-dimensional Brownian motion under the so called forward-neutral measure $\mathbb{Q}$ [5], the matrix process σ takes values in a closed bounded set $\Sigma \subset M(n, n, \mathbb{R})$, and we use the notation $\bar{s} = \mathrm{diag}\,(s)$ for $s \in \mathbb{R}^n$. We then consider an agent who wants to price and hedge a European contingent claim whose payoff is a continuous deterministic function $h(\cdot)$ with polynomial growth, calculated in S_T. Since the market could be incomplete because of the stochastic volatility σ and the agent is not able to hedge the volatility, he chooses to hedge the option by using the superhedging approach. In order to build a superstrategy, he fixes the price of the option as $C_t = C(t, S_t)$ and builds a self-financing portfolio consisting of a quantity $\Delta_t^i = \frac{\partial C}{\partial S_i}(t, S_t)$ of the asset $S^i, i = 1, \dots, n$, where $C(t, S)$ is a solution of the following nonlinear PDE

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \mathrm{tr}\, \left(D^2 C(t,s) (\bar{s}\gamma)(\bar{s}\gamma)^* \right) = 0 \ , & t \in [0,T), s \in \mathbb{R}_+^n \ , \\ C(T,s) = h(s) \ , & s \in \mathbb{R}_+^n \ , \end{cases} \tag{1}$$

called Black-Scholes-Barenblatt (BSB) equation by analogy with [1], where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s), \dots, \frac{\partial C}{\partial s_n}(t,s))$, and $D^2C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (1) is a Hamilton-Jacobi-Bellman equation, and it is linked to a stochastic control problem that has a nice financial interpretation. In fact, the agent could build a “subjective” model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \, dW_t \ ,$$

where the state variable S^γ has to be controlled by the “subjective” volatility γ in order to maximize the payoff function

$$\mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} h(S_T^\gamma) \mid \mathcal{F}_t \right] \ ,$$

corresponding to the no-arbitrage price of the contingent claim at time t. In this way, the agent wants to maximize in γ his payoff thus protecting himself against “the worst possible case”, and Eq. (1) is the HJB equation corresponding to this optimal control problem. The central result of this paper is Theorem 2, where we characterize the superstrategies in terms of a quantity R_t, that could be interpreted as a consumption rate, and we prove that the price-hedge strategy above is the cheapest Markov superstrategy in a sufficiently large class. Equation (1) hides a simple problem, namely the maximum problem which appears in it. This problem is significant because the agent must solve it in order to implement the model in the right way. We find some necessary and sufficient conditions for the existence of a solution to the maximum problem. Our conditions provide a nice geometrical interpretation of the problem. Moreover we find that under particular conditions it is possible to reduce the nonlinear BSB equation to a linear Black-Scholes (BS) equation. We show also that it is possible to obtain as a particular case the results of [1] and [6]. Finally we analyse some examples: Margrabe’s exchange option, for which we find that the BSB equation is always reduced to a BS equation; a European option on the geometric mean of two assets, which gives us the feeling of the behaviour of a non-convex payoff, and last, an example of a possible set Σ, suggested by a situation of stochastic correlation between two assets, where we find sufficient conditions in order to reduce the BSB equation to a BS equation.

The papers in which this kind of approach to stochastic volatility models appears for the first time are [1] and [6] for the 1-dimensional case, and [13] for the multidimensional one. In [6] the problem is widely treated in the case of European options having a payoff h convex in the asset price. For this kind of options, they succeed in showing a superstrategy depending on the solution of a particular BS equation, obtained by dominating the stochastic volatility with a deterministic function of the price. However, options are traded in the markets that have genuine non-convex payoffs, such as call-spread options, that are difference between two calls with different strike prices and same maturity. As noticed in [1] the approach in [6] is not generalizable to this kind of payoffs. To this end in [1] the authors introduce the BSB equation in 1 dimension. In their case the set Σ is a closed interval $[\sigma_{\min}, \sigma_{\max}]$ in the real line, with $0 < \sigma_{\min} < \sigma_{\max} < \infty$. They prove a result analogous to our Theorem 2 in dimension 1. Since for a long time options on several assets have a theoretical treatment (see [10], [14] and [15] for some examples), we treated the case of European options on several assets, as Lyons did in [13]. While Lyons uses stochastic integration without probability, in our work we use the theory of stochastic control. We also present some examples of contingent claims on several assets. For the sake of simplicity, we have not included the case of interest rate derivatives as caps, floors or swaptions, which are a typical example of multiasset options, because in the case of interest rate derivatives other topics would deserve our attention. This case is however being studied by one of the two authors. Moreover, we choose not to study problems such as existence, uniqueness and regularity of the solution of Eq. (1) in detail, but we choose to include a section presenting in a qualitative way the results obtained in another work by one of the two authors [9]. Finally, we want to point out that the BSB equation appears also in other related problems, for example when one has portfolio constraints [4].

The paper is organized as follows: in Section 2 we present the problem and give the characterization theorem; in Section 3 we present some results on the BSB equation; in Section 4 we give sufficient conditions to solve the optimization problem, and we show how it is possible to reduce the BSB equation to a BS equation when the optimization problem has a constant solution; in Section 5 we obtain as particular cases the results of [1] and [6]; in Section 6 we apply our results to Margrabe's exchange option, and in Section 7 we do the same for a call option on the geometric mean of two assets; in Section 8 we present a stochastic correlation problem, in which there are explicit solutions for the BSB equation.

We wish to thank Vincent Lacoste for introducing us to the works [1] and [6], from which we started, Dario Bini and Mauro Nacinovich for their deep understanding of the geometry of the optimization problem, and Claude Martini who suggested that we study the stochastic correlation problem in Section 8. We also wish to thank an anonymous referee who pointed out many ambiguities in the first draft of this manuscript.

2. The model

We suppose that there exists a riskless asset M and n risky assets S^i, $i = 1, \ldots, n$ in the market. We make the usual assumptions that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t$ represents the information available up to time t and that M and S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Besides, we assume that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is called **forward-neutral probability** [5], such that the value of the riskless asset M remains constantly equal to 1 through time, and the dynamics of the assets S^i under $\mathbb{Q}$ are the following:

$$dS_t^i \quad = \quad S_t^i \langle \sigma_t^i, dW_t \rangle \tag{2}$$

where $(W_t)_t$ is a n-dimensional $\mathbb{Q}$-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^n$ and σ^i is a n-dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\Sigma)$, where Σ is a closed bounded set in the space of $n \times n$ real matrices $M(n, n, \mathbb{R})$ and $\mathcal{A}(\Sigma)$ (which we call set of **admissible volatilities**) is the set of Σ-valued processes progressively measurable with respect to $(\mathcal{F}_t)_t$.

We can write the dynamics of the risky assets in a more compact vectorial notation as

$$dS_t = \bar{S}_t \sigma_t \; dW_t \ ,$$

where for a given $s \in \mathbb{R}^n$, we put $\bar{s} = \mathrm{diag}\ (s)$, the $n \times n$ diagonal matrix having the i-th element equal to s_i. Now we consider an agent in the market who wants to create and sell a European derivative asset with payoff $h(S_T) = h(S_T^1, \ldots, S_T^n)$, where h is a measurable function having polynomial growth.

Remark 1. We do not assume that the market is complete. In particular, the filtration $(\mathcal{F}_t)_t$ could be strictly larger than the one generated by W or the one generated by S. In this case we have a genuine stochastic volatility model and the market is incomplete. Moreover, the assumption that the interest rate is zero can be achieved by a so-called change of numeraire, that is, by expressing the prices of all the assets in the market in units of a zero-coupon bond maturing at time T (for more details, see [5]).

In order to price and hedge the asset, the agent fixes a price C_t for the asset at time t with the requirement that $C_T = h(S_T)$ and builds a self-financing portfolio Π, holding η_t units of the money market account M and Δ_t^i units of the asset S^i at time t. We indicate with Π_t the value of the portfolio at time t, defined by

$$\Pi_t = \eta_t + \langle \Delta_t, S_t \rangle \ , \tag{3}$$

where $\Delta_t = (\Delta_t^1, \ldots, \Delta_t^n)$. We say that the portfolio is **admissible** if Π is a supermartingale: in this way if the portfolio is admissible and $\Pi_0 = 0$, then we have $\mathbb{E}[\Pi_T] \leq 0$, so there are not arbitrage opportunities in the market. We also say that the portfolio is **self-financing** if Π follows the dynamic

$$\begin{cases} d\Pi_t = \langle \Delta_t, dS_t \rangle \ , \\ \Pi_0 = C_0 \ . \end{cases} \tag{4}$$

If the portfolio is self-financing, the process Δ is sufficient to characterize it, and $\eta_t = \Pi_t - \langle \Delta_t, S_t \rangle$.

The market is not complete because of the stochastic volatility σ, so the agent chooses to hedge the claim using the superhedging approach. To this end, we define the **tracking error** as

$$e_t = \Pi_t - C_t \ ,$$

$e_0 = 0$ by definition of Π. Intuitively, the tracking error gives the error made by the agent in estimation, or better, the difference between the hedging portfolio held by the agent and the option price. If the portfolio is admissible and self-financing, we say that (C, Δ) is a

- **superhedging strategy**, or simply **superstrategy** if e is a non decreasing process
- **subhedging strategy**, or simply **substrategy**, if e is a non increasing process
- **hedging strategy** if e is identically equal to zero.

The sense of these definitions is the following: if one builds a superstrategy, one can successfully hedge a short position in the contingent claim C_t. In particular with a superstrategy we have that $e_T \geq 0$, that is $\Pi_T \geq C_T = h(S_T)$, so the portfolio succeeds in overhedging the contingent claim. We notice that superstrategies and substrategies are good candidates to be arbitrages: in fact if we have (say) $\Pi_T \geq C_T$ $\mathbb{Q}$-a.s. and $\Pi_T > C_T$ with positive probability, then the agent succeeds in making a profit with no initial endowment by selling the option at price C_0 and buying the superreplication portfolio at price $\Pi_0 = C_0$ at time 0. This means that the value C_t in a superstrategy has to be interpreted as an arbitrage upper bound for the price of the claim at time t. Similarly, the value C_t in a substrategy has to be interpreted as a lower bound.

As already outlined by [1], [6] and [13], in order to have a superstrategy, a natural procedure is this: the agent fixes the price C_t of the option and the quantities Δ_t of the risky assets in the hedging portfolio Π at time t as

$$C_t = C(t, S_t) \ , \qquad \Delta_t^i = \frac{\partial C}{\partial s_i}(t, S_t) \ ,$$

where C is the solution of the following partial differential equation:

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \mathrm{tr}\ (D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*) = 0 \ , & t \in [0,T), s \in \mathbb{R}_+^n \ , \\ C(T,s) = h(s) \ , & s \in \mathbb{R}_+^n \ , \end{cases} \tag{5}$$

where $DC(t,s) = (\frac{\partial C}{\partial s_1}(t,s), \ldots, \frac{\partial C}{\partial s_n}(t,s))$, and $D^2 C(t,s) = (\frac{\partial^2 C}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (5), as in [1], will be called from now on the **Black-Scholes-Barenblatt equation** for h. Moreover, in order to have a substrategy it is sufficient to replace the maximum operator in Eq. (5) with a minimum.

This choice has a stochastic control interpretation: we assume that the agent does not know the volatility σ, but in order to hedge the claim he can use the model

$$dS_t^\gamma = \bar{S}_t^\gamma \gamma_t \ dW_t \ , \tag{6}$$

where the process $\gamma \in \mathcal{A}(\Sigma)$ is another admissible volatility. We can interpret the γ as a control, corresponding to the subjective volatility of the agent, that he uses in order to decide his strategy. If the market were complete and the dynamics were given by Eq. (6), then the arbitrage free price of the European contingent claim would be

$$C_t^{\mathbb{Q}',\gamma} = \mathbb{E}_{\mathbb{Q}'}\left[h(S_T^\gamma) \mid S_t^\gamma\right] . \tag{7}$$

Since the market is not complete, we are not sure that the contingent claim defined by h is attainable. In particular, we have two sources of uncertainty, namely the particular forward-neutral measure $\mathbb{Q}$ and the volatility γ. Since the agent does not know either of these objects, he associates the price of the claim with the interval

$$\left\{ \mathbb{E}_{\mathbb{Q}'}\left[e^{-r(T-t)} h(S_T^\gamma) \middle| \mathcal{F}_t \right] \;\middle|\; \mathbb{Q}' \text{ e.f.m.}, \gamma \in \mathcal{A}(\Sigma) \right\} ,$$

where e.f.m. means that $\mathbb{Q}'$ is a generic forward measure such that the fictitious prices S^γ obey an equation of the kind (6) under $\mathbb{Q}'$ and W is a $\mathbb{Q}'$-Brownian motion. This is an interval in the real line, called the **set of admissible prices**. However, since $\gamma \in \mathcal{A}(\Sigma)$, for all the forward measures $\mathbb{Q}$ and $\gamma \in \mathcal{A}(\Sigma)$ we have

$$C_{\mathbb{Q}}^-(t, S_t^\gamma) \le C_t^{\mathbb{Q},\gamma} \le C_{\mathbb{Q}}^+(t, S_t^\gamma) ,$$

where

$$C_{\mathbb{Q}}^-(t,s) = \inf_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \middle| S_t^\gamma = s\right] \tag{8}$$

and

$$C_{\mathbb{Q}}^+(t,s) = \sup_{\gamma \in \mathcal{A}(\Sigma)} \mathbb{E}_{\mathbb{Q}}\left[h(S_T^\gamma) \middle| S_t^\gamma = s\right] \tag{9}$$

are the value functions of two optimal control problems, having as payoff function the right-hand side of Eq. (7). In particular, under our assumptions, $C_{\mathbb{Q}}^+$ is the unique viscosity solution of Eq. (5) (see Section 3) and does not depend on the particular choice of the measure $\mathbb{Q}$ (so we can drop the subscript in $\mathbb{Q}$). So we have that $[C^-(t, S_t), C^+(t, S_t)]$ defines the interval of admissible arbitrage prices for the claim at time t, in the sense that if the price of the claim lies outside this interval, then it is possible to build an arbitrage in the market. This can be interpreted as follows: when the agent decides the price of the claim, he wishes to protect himself against the worst possible outcome, and he chooses the corresponding price. Moreover, we can interpret the γ as a control, corresponding to the subjective volatility of the agent used to decide his strategy.

We notice that the substrategy case is in some sense symmetric to that of the superstrategy. This has a simple mathematical explanation: in fact, if C is the solution of the BSB equation with final condition h, then $-C$ is the solution of the BSB equation with min instead of max, and final condition $-h$. In other words, if C defines a superstrategy for the claim defined by h, then $-C$ defines a substrategy for the claim defined by $-h$. This allows us to analyse only one of the two cases, and to obtain results for the other one automatically. For this reason, in the rest of the paper we will analyse only the superstrategy case, it being implicit that analogous results hold also for substrategies.

We now exhibit a superstrategy. We indicate with $C_p^{1,2}((0,T)\times\mathbb{R}_+^n)$ the space of functions C that are continuous and with polynomial growth on $(0,T)\times\mathbb{R}_+^n$ together with their first derivative in t and first and second derivatives in s.

Theorem 2. *If we restrict ourselves to the case* $C_t = C(t, S_t)$*, with* $C \in C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$*, then:*

i) (C, Δ) *is a superstrategy if and only if* $\Delta_t = DC(t, S_t)$ *and*

$$R_t = -\frac{\partial C}{\partial t}(t, S_t) - \frac{1}{2}\mathrm{tr}\ D^2 C(t, S_t)(\bar{S}_t \sigma_t)(\bar{S}_t \sigma_t)^*$$

is a.s. non negative for all $t \in [0, T]$*;*

ii) if C *is the solution of Eq. (5) and* $\Delta_t = DC(t, S_t)$*, then* (C, Δ) *is a superstrategy. Moreover there does not exist another superstrategy* (C', Δ') *such that* $C'_t < C(t, S_t^\gamma)$ *with positive probability for some* t *and for all* $\gamma \in \mathcal{A}(\Sigma)$*.*

Proof. The evolution of $(e_t)_t$ is the following:

$$\begin{aligned} de_t &= \langle \Delta_t, dS_t \rangle - \frac{\partial C}{\partial t}(t, S_t)\ dt - \langle DC(t, S_t), dS_t \rangle - \\ &\quad -\frac{1}{2}\mathrm{tr}\ \left(D^2 C(t, S_t)(\bar{S}_t \sigma_t)(\bar{S}_t \sigma_t)^* \right)\ dt = \\ &= R_t\ dt + \langle \Delta_t - DC(t, S_t), dS_t \rangle\ . \end{aligned}$$

Now, in order for the process e to be increasing, it must have finite variation, so we must have $\Delta_t = DC(t, S_t)$. Furthermore, if this condition holds then e will be increasing if and only if $R_t \geq 0$ $\mathbb{Q}$-a.s. $\forall t$. Conversely, if these two conditions are satisfied, then e is increasing. Finally, Δ is admissible. In fact, since $C \in C_p^{1,2}$, we have that

$$\mathbb{E}\left[\int_0^T \|\sigma_t^* \bar{S}_t \Delta_t\|^2\ dt\right] \leq M \mathbb{E}\left[\int_0^T \|S_t\|^2 C^2 (1 + \|S_t\|^m)^2\ dt\right]$$

for some C and m, where $M = \sup\{\lambda \mid \|\gamma^* x\| \leq \lambda \|x\| \quad \forall x \in \mathbb{R}^n, \gamma \in \Sigma\}$. Since for all $m \geq 2$ we have $\mathbb{E}[\|S_t\|^m] \leq B_m(1 + \|S_0\|^m)$ for a suitable B_m (see [11]), we have $\mathbb{E}[\int_0^T \|\sigma_t^* \bar{S}_t \Delta_t\|^2\ dt] < +\infty$, so $\Pi_t = \int_0^t \langle \Delta_u, dS_u \rangle = \int_0^t \langle \sigma_u^* \bar{S}_u \Delta_u, dW_u \rangle$ is a martingale.

In order to prove (ii), we notice that under its assumptions we have that

$$R_t = \frac{1}{2}\left(\max_{\gamma \in \Sigma} \mathrm{tr}\ \left(D^2 C(t, S_t)(\bar{S}_t \gamma)(\bar{S}_t \gamma)^* \right) - \mathrm{tr}\ \left(D^2 C(t, S_t)(\bar{S}_t \sigma_t)(\bar{S}_t \sigma_t)^* \right) \right) \geq 0\ .$$

Moreover, C is a classical solution of Eq. (5), so it is equal to the value function (9). Now we assume that there exists another superstrategy (C', Δ') such that $C_0 = \Pi_0 < C(0, S_0)$. We put $\varepsilon = C(0, S_0) - C_0$; then Π is a supermartingale, so $\Pi_0 \geq \mathbb{E}[\Pi_T]$. On the other hand, we also have that

$$C(0, S_0) = \max_{\gamma \in \Sigma} \mathbb{E}[h(S_t^\gamma)]\ .$$

We take $\bar{\gamma}$ such that $\mathbb{E}[h(S_t^{\bar{\gamma}})] > C(0, S_0) - \varepsilon = \Pi_0$; then $\mathbb{E}[h(S_t^{\bar{\gamma}})] > \mathbb{E}[\Pi_T]$. This means that $\mathbb{P}\{h(S_T^{\bar{\gamma}}) > \Pi_T\} > 0$, and we have the result. □

We have now obtained a characterization theorem for superstrategies. In fact, provided the value function $C \in C_p^{1,2}$, the theorem gives the minimal superstrategy, in the sense that there does not exist a cheaper strategy that succeeds in giving an increasing tracking error for all $\gamma \in \mathcal{A}(\Sigma)$. Since the agent does not know the true volatility σ, he must protect himself against all the possible volatilities $\gamma \in \mathcal{A}(\Sigma)$ and this is the best result he can expect.

The quantity R_t can be seen as an instantaneous rate of consumption obtained by the superstrategy of the agent. A discussion of the financial interpretation of R can be found in [13].

An analogous of Theorem 2 holds for the case of a substrategy, provided we reverse the inequalities and substitute the max operator in Eq. (5) with a min.

3. The BSB equation

We have seen that in order to have a Markov superstrategy, we need a $C^{1,2}$ solution of Eq. (5). To this end, we make a brief survey of the results existing in the literature about nonlinear parabolic equations. We do not wish to give rigorous proofs of our assertions, but only to offer some ideas of how things can be done. A rigorous treatment of the problem is the topic of a forthcoming work [9] by one of the two authors.

The first step is to notice that the value function is a solution of Eq. (5) in a weaker sense, namely in the sense of the theory of viscosity solutions. This theory, developed by Crandall and Lions in a series of papers in the early '80s, allows for powerful existence and uniqueness theorems, whose proofs are much easier than the corresponding ones in the classical theory (see [3] for a survey). Moreover, there are interesting links between optimal control problems and viscosity solutions of the corresponding HJB equations: in particular, under technical assumptions, the value function is the unique viscosity solution of the HJB equation (see [8]). In our situation, we have the following result, whose proof can be found in [9].

Theorem 3. *If h has polynomial growth, then the value function C defined by Eq. (9) is a viscosity solution of Eq. (5) in $[0,T] \times \mathbb{R}_+^n$. Moreover, it is the unique viscosity solution having polynomial growth that satisfies the boundary condition $C(T,s) = h(s)$ for all $s \in \mathbb{R}_+^n$.*

However, a priori viscosity solutions are only continuous, so if we want to use Theorem 2 with a viscosity solution, we also need to prove that the solution is smooth enough to apply Itô's formula. The main reference in this topic is Wang's work [16]. In order to use his results, we have to make a change of variable. We put $y_i = \log s_i$. Then the BSB equation becomes:

$$\begin{cases} \dfrac{\partial \tilde{C}}{\partial t}(t,y) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \operatorname{tr}\left((D^2 \tilde{C}(t,y) - \operatorname{diag}(D\tilde{C}(t,y)))\gamma\gamma^* \right) = 0 \ , & t \in [0,T) \ , y \in \mathbb{R}^n \ , \\ \tilde{C}(T,y) = \tilde{h}(y) \ , & y \in \mathbb{R}^n \ , \end{cases} \tag{10}$$

where $\tilde{C}(t,y) = C(t, (\mathrm{e}^{y_i})_i) = C(t,s)$. We then say that Eq. (10) is **uniformly parabolic** if there exist real numbers $M > m > 0$ such that for all $\gamma \in \Sigma$ and $\xi \in \mathbb{R}^n$,

$$m\|\xi\|^2 \le \langle \gamma\gamma^* \xi, \xi \rangle \le M\|\xi\|^2 .$$

Since $\langle \gamma\gamma^*\xi, \xi\rangle = \langle \gamma^*\xi, \gamma^*\xi\rangle = \|\gamma^*\xi\|^2$, the definition can be rewritten in this way: there exist real numbers $M' > m' > 0$ such that for all $\gamma \in \Sigma$ and $\xi \in \mathbb{R}^n$,

$$m'\|\xi\| \le \|\gamma^*\xi\| \le M'\|\xi\| .$$

This implies that if there exists a $\gamma \in \Sigma$ which is not invertible, then Eq. (6) is not uniformly parabolic. In fact if we take $\xi \in \ker \gamma^*$, then $\|\gamma^*\xi\| = 0$. We show that also the converse is true.

Lemma 4. *If Σ is closed and bounded in $M(n, n, \mathbb{R})$, then Eq. (10) is uniformly parabolic if and only if $\Sigma \subseteq \mathrm{GL}(n, \mathbb{R})$, where*

$$\mathrm{GL}(n, \mathbb{R}) = \{\gamma \in M(n, n, \mathbb{R}) \mid \det \gamma \neq 0\} .$$

Proof. As seen above, if $\exists \gamma \in \Sigma$ such that $\gamma \notin \mathrm{GL}(n, \mathbb{R})$, then Eq. (10) is not uniformly parabolic. Conversely, if $\Sigma \subseteq M(n, n, \mathbb{R})$, then $\gamma\gamma^*$ is positive definite for all $\gamma \in \Sigma$. Moreover, the function that goes from γ to the least eigenvalue of $\gamma\gamma^*$ is continuous, so it has a minimum $m' > 0$ in the compact set Σ; also the function going from γ to the greater eigenvalue of $\gamma\gamma^*$ is continuous, so it has a maximum $M' > 0$ in Σ. Thus the result follows. □

This allows us to state the following theorem.

Theorem 5. *If Eq. (10) is uniformly parabolic, and we call $\tilde{C}$ its viscosity solution, then there exists $\alpha \in (0, 1]$ such that $\tilde{C} \in C^{1,2,\alpha}(\mathbb{R}^n)$.*

This is a particular case of a general result in [16], and means that $\tilde{C}$ has second derivatives in s that are α-Hölder continuous. This means that also $C(t, s_1, \ldots, s_n) = \tilde{C}(t, \log s_1, \ldots, \log s_n)$ is C^2, so we can apply Itô's formula. The final conclusion is that if the set of volatilities Σ is a closed bounded set in $\mathrm{GL}(n, \mathbb{R})$, then there exists a Markov superstrategy, and it is given by the solution of Eq. (5).

Remark 6. Uniform parabolicity of Eq. (10) is not always needed to have regularity of solutions. In fact there are significant cases (for example, the one treated in Section 7) in which Eq. (10) is not uniformly parabolic, but it still has $C^{1,2}$ solutions. A work [9] on this topic has been carried out by one of the authors.

4. The optimization problem

We notice that Eq. (5) contains an optimization problem. The agent has to solve this problem if he wants to solve the BSB equation numerically, and if he wants to have a view of what could be the "worst case" against him. Moreover, it turns out that in some cases the problem gives a simple way to pass from the BSB equation to a Black-Scholes equation, which is simpler both from a theoretical point of view as well as from the numerical side. In any case, this problem is significant only if $C \in C^{1,2}$, so in the following we will make that assumption.

The problem can be seen as an optimization problem of a bilinear form in the spaces of $n \times n$ real matrices, so we can characterize the solution and if Σ is smooth enough we can solve the problem explicitly.

The optimization problem is

$$\max_{\gamma\in\Sigma} F_{t,s}(\gamma) \ , \tag{11}$$

where $F_{t,s} : \Sigma \to \mathbb{R}$ is defined by

$$F_{t,s}(\gamma) = \mathrm{tr}\ \left(\bar{s}D^2C(t,s)\bar{s}\gamma\gamma^*\right) = \mathrm{tr}\ \left(A_{t,s}\gamma\gamma^*\right) \ , \tag{12}$$

where $A_{t,s} = \bar{s}D^2C(t,s)\bar{s}$ and C is the solution of Eq. (5). We will often write F and A instead of $F_{t,s}$ and $A_{t,s}$ if there is no risk of ambiguity. This is an optimization problem of a bilinear form in the closed bounded set $\Sigma \subset M(n,n,\mathbb{R})$. This allows us to make a first statement.

Lemma 7. *If $0 \notin \Sigma$, then F assumes the maximum on $\partial\Sigma$.*

Proof. Since F is continuous and Σ is a compact set, then F has a maximum $\bar{\gamma}$ in Σ. Let's suppose that $F(\bar{\gamma}) > 0$ and $\bar{\gamma}$ is in $\overset{\circ}{\Sigma}$. Then there exists an $\varepsilon > 0$ such that $(1+\varepsilon)\bar{\gamma} \in \Sigma$, and

$$F((1+\varepsilon)\bar{\gamma}) = (1+\varepsilon)^2 F(\bar{\gamma}) > F(\bar{\gamma}) \ ,$$

so $\bar{\gamma}$ is not maximum, which is a contradiction. In an analogous way, if $F(\bar{\gamma}) < 0$ and $\bar{\gamma}$ is in $\overset{\circ}{\Sigma}$, then we can repeat the above argument with $(1-\varepsilon)\,\bar{\gamma}$, reaching a contradiction. Finally if $F(\bar{\gamma}) = 0$ we can find a $\gamma^* \in \partial\Sigma$ such that $F(\gamma^*) = 0$, so we have the result. □

Luckily, it would be unrealistic to include 0 in Σ, because if $0 \in \Sigma$, then there is the possibility of a null volatility in all the assets. If $0 \notin \Sigma$, we can reformulate the problem in this way:

$$\max_{\gamma\in\partial\Sigma} F(\gamma) \ . \tag{13}$$

Now we present a first-order condition, that has to be satisfied if the set Σ has smooth boundary, whose proof follows immediately from an application of the Lagrange multiplier method.

Proposition 8. *If $\partial\Sigma$ is a smooth (C^1) curve defined by $G(\gamma) = 0$ and $\bar{\gamma}$ is a maximum for F, then $\exists\lambda \in \mathbb{R}$ such that $2A\bar{\gamma} - \lambda DG(\bar{\gamma}) = 0$, where DG indicates the gradient with respect to γ.*

Remark 9. We have seen that if $\partial\Sigma$ is defined by $G(\gamma) = 0$, with $G \in C^1(M(n,n,\mathbb{R}),\mathbb{R})$, then $\bar{\gamma}$ is a solution of the system:

$$\begin{cases} 2A\gamma - \lambda DG(\gamma) = 0 \ , \\ G(\gamma) = 0 \ . \end{cases}$$

In particular, $\bar{\gamma}$ depends continuously on A. Viceversa, if $\partial\Sigma$ is not regular, the dependence of $\bar{\gamma}$ on A may not necessarily be continuous.

An important case is when the optimization problem has a constant solution. In this case, we can reduce the non linear BSB equation to a standard Black-Scholes equation.

Proposition 10. *If $C \in C^{1,2}([0,T) \times \mathbb{R}^n)$ is a solution of the Black-Scholes equation*

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\mathrm{tr}\ (D^2 C(t,s)(\bar{s}\bar{\gamma})(\bar{s}\bar{\gamma})^*) = 0, & t \in [0,T), s \in \mathbb{R}^n_+\ , \\ C(T,s) = h(s), & s \in \mathbb{R}^n_+\ , \end{cases} \tag{14}$$

with $\bar{\gamma} \in \Sigma$, such that the problem (11) attains its maximum in $\bar{\gamma}$ for all $(t,s) \in [0,T) \times \mathbb{R}^n_+$, then C is also a solution of Eq. (5).

Proof. The maximum problem (11) has solution

$$\mathrm{tr}\ (D^2 C(t,s)(s\bar{\gamma})(s\bar{\gamma})^*) = \max_{\gamma \in \Sigma} \mathrm{tr}\ (D^2 C(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*)\ ,$$

so the conclusion follows. □

This proposition is very useful for reducing the BSB equation, for which explicit solutions are very rare, to a BS equation, for which explicit solutions are common. We will see an example of this case in the next sections.

5. The one-dimensional case

Now we analyse in more detail the case in which $n = 1$. This was the first case studied in the literature. The main references are [1] and [6]. The first thing to notice is that the typical example of Σ in this case is $\Sigma = [\sigma_{\min}, \sigma_{\max}]$, with $0 < \sigma_{\min} < \sigma_{\max} < \infty$. For this reason, throughout this section we will assume that $n = 1$ and $\Sigma = [\sigma_{\min}, \sigma_{\max}]$.

We obtain the following results, which are already present in [1].

Lemma 11. *The problem (11) has solution*

$$\sigma^+\left(\frac{\partial^2 C}{\partial s^2}(t,s)\right) = \begin{cases} \sigma_{\max} & \text{if } \dfrac{\partial^2 C}{\partial s^2}(t,s) \geq 0 \\ \sigma_{\min} & \text{if } \dfrac{\partial^2 C}{\partial s^2}(t,s) < 0 \end{cases}$$

so the BSB equation becomes

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,s) + \dfrac{1}{2}\sigma^+\left(\dfrac{\partial^2 C}{\partial s^2}(t,s)\right)^2 s^2 \dfrac{\partial^2 C}{\partial s^2}(t,s) = 0, & t \in [0,T), s \in \mathbb{R}^n_+ \\ C(T,s) = h(s), & s \in \mathbb{R}^n_+ \end{cases}$$

Proof. In this case, $F_{t,s}(\gamma) = s^2 \frac{\partial^2 C}{\partial s^2}(t,s)\gamma^2$, so the conclusion follows. □

A consequence is that in dimension 1 a suitable BS equation in some cases can provide a solution of the BSB equation, as we shall now see.

Theorem 12. *If $n = 1$ and h is convex, then the solution to the BS equation (14) with volatility $\sigma_{\max}$ is also solution of the BSB equation (5), thus giving a superstrategy.*

Proof. The solution C of Eq. (14) with volatility $\sigma_{\max}$ is convex, so if we calculate F in problem (12), then the problem has solution identically equal to $\sigma_{\max}$; this means that C is also solution of Eq. (5). □

6. Margrabe's exchange option

In the next two sections we consider two examples of claims depending on two primary assets, in order to have an idea of how things change when one deals with multiasset payoffs.

First we consider Margrabe's exchange option (see [14]), whose final payoff is

$$h(s) = (s_1 - \lambda s_2)^+ .$$

We try to apply the results of Section 4 to this option. To this end, for a generic $\gamma \in \Sigma$ we calculate the Black-Scholes price $C(t, s_1, s_2)$ of the exchange option, which is the solution of Eq. (14) when we substitute $\bar{\gamma}$ with

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} . \tag{15}$$

By Proposition 10, if the Black-Scholes price is such that the optimization problem (11) attains its maximum at γ for all $t \in [0, T)$, $s \in \mathbb{R}^2_+$, then $C(t, s_1, s_2)$ is also a solution of the BSB equation (5).

The Black-Scholes price of this option is:

$$C(t, s_1, s_2) = s_1 N(d_1) - \lambda s_2 N(d_2) , \tag{16}$$

where

$$d_1 = \frac{1}{\Gamma\sqrt{T-t}} \ln\left(\frac{s_1}{\lambda s_2}\right) + \frac{1}{2}\Gamma\sqrt{T-t} , \quad d_2 = d_1 - \Gamma\sqrt{T-t} ,$$

$$\Gamma = \sqrt{(\gamma_{11} - \gamma_{21})^2 + (\gamma_{12} - \gamma_{22})^2}$$

and N is the cumulative distribution function of a centered reduced Gaussian random variable:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} \mathrm{e}^{-\frac{x^2}{2}} \, dx$$

(for a proof, see [14]). In order to apply Proposition 10, we calculate the matrix $A_{t,s}$. The first derivatives are

$$\frac{\partial C}{\partial s_1} = N(d_1) , \quad \frac{\partial C}{\partial s_2} = -\lambda N(d_2) .$$

The second derivatives are

$$\frac{\partial^2 C}{\partial s_1^2} = \frac{\partial d_1}{\partial s_1} N'(d_1) = \frac{1}{s_1 \Gamma\sqrt{T-t}} N'(d_1) ,$$

$$\frac{\partial^2 C}{\partial s_2^2} = \lambda \frac{\partial d_2}{\partial s_2} N'(d_2) = \lambda \frac{1}{s_2 \Gamma\sqrt{T-t}} N'(d_2) ,$$

because

$$\frac{\partial d_i}{\partial s_1} = \frac{1}{s_1 \Gamma \sqrt{T-t}} \ , \quad \frac{\partial d_i}{\partial s_2} = -\frac{1}{s_2 \Gamma \sqrt{T-t}} \ , \quad i = 1, 2 \ .$$

We have two ways to calculate the cross derivative:

$$\frac{\partial^2 C}{\partial s_1 \partial s_2} = \frac{\partial d_1}{\partial s_2} N'(d_1) = -\lambda \frac{\partial d_2}{\partial s_1} N'(d_2) = -\frac{N'(d_1)}{s_2 \Gamma \sqrt{T-t}} = -\frac{\lambda N'(d_1)}{s_2 \Gamma \sqrt{T-t}} \ .$$

Since $C \in C^{1,2}$, from this we get that $s_1 N'(d_1) = \lambda s_2 N'(d_2)$. Thus we have that

$$A_{t,s} = \bar{s} \cdot D^2 C(t,s) \cdot \bar{s} = \begin{pmatrix} s_1^2 \frac{\partial^2 C}{\partial s_1^2} & s_1 s_2 \frac{\partial^2 C}{\partial s_1 s_2} \\ s_1 s_2 \frac{\partial^2 C}{\partial s_1 s_2} & s_2^2 \frac{\partial^2 C}{\partial s_2^2} \end{pmatrix} = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ .$$

We notice that $A_{t,s}$ is proportional $\forall (t,s) \in [0,T) \times \mathbb{R}^2_+$ to the constant matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ .$$

This means that the solution to the problem (11) is constant. Hence we easily obtain the following proposition.

Proposition 13. *If $\bar{\gamma}$ is the maximum of $\|\gamma_1 - \gamma_2\|_{\mathbb{R}^2}$ on Σ, where γ_1, γ_2 are the rows of γ, then (16) is solution of Eq. (5).*

Proof. We have that $\forall a, b, c \in \mathbb{R}$,

$$\mathrm{tr} \left(A_{t,s} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} (a - 2b + c) \ .$$

so by easy calculations we obtain that

$$\mathrm{tr} \ (A_{t,s} \gamma \gamma^*) = \frac{s_1 N'(d_1)}{\Gamma \sqrt{T-t}} \|\gamma_1 - \gamma_2\|^2_{\mathbb{R}^2} \ .$$

The result follows then from Proposition 10. □

7. An example of a non-convex payoff: option on the geometric mean

We consider an option whose final payoff is the following:

$$h(s) = (\sqrt{s_1 s_2} - K)^+ \ .$$

As in the previous section, we try to apply the results of Section 4 to this option. To this end, we first calculate the Black-Scholes price in the case in which the dynamics is given by

$$dS_t = \bar{S}_t \gamma \ dW_t \ , \tag{17}$$

where γ is deterministic and given by Eq. (15).

The dynamics of $\sqrt{S_1 S_2}$ is

$$d\sqrt{S_1(t)S_2(t)} = \sqrt{S_1(t)S_2(t)}\left(\frac{1}{2}\langle\gamma_1+\gamma_2, dW(t)\rangle + \mu\ dt\right) ,$$

where $\mu = -\frac{1}{8}\|\gamma_1-\gamma_2\|^2$. We can interpret $\sqrt{S_1S_2}$ as an asset having instantaneous rate of return equal to μ, so we can apply the Black-Scholes formula and obtain the price of the option as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\left(\sqrt{S_1(T)S_2(T)}-K\right)^+\right] &= \mathbb{E}_{\mathbb{Q}}\left[\left(\sqrt{S_1(T)S_2(T)}-K\right)^+\right] = \\ &= \mathrm{e}^{\mu(T-t)}\left(\sqrt{S_1(t)S_2(t)}N(d_1)-K\mathrm{e}^{-\mu(T-t)}N(d_2)\right) ,\end{aligned}$$

where

$$d_1 = \frac{1}{\tilde{\gamma}_t}\ln\frac{\sqrt{S_1(t)S_2(t)}}{Ke^{-\mu(T-t)}}+\frac{1}{2}\tilde{\gamma}_t , \quad d_2 = d_1-\tilde{\gamma}_t , \quad \tilde{\gamma}_t = \frac{1}{2}\|\gamma_1+\gamma_2\|\sqrt{T-t} .$$

The price of the option is then:

$$C(t,s_1,s_2) = A\left(\sqrt{s_1s_2}N(d_1)-Ke^{-\mu(T-t)}N(d_2)\right) , \tag{18}$$

where $A = \mathrm{e}^{\mu(T-t)}$. Now we calculate $A_{t,s}$. The first derivatives of C are

$$\frac{\partial C}{\partial s_1} = \frac{A\sqrt{s_2}}{2\sqrt{s_1}}N(d_1) , \quad \frac{\partial C}{\partial s_2} = \frac{A\sqrt{s_1}}{2\sqrt{s_2}}N(d_1)$$

The second derivatives of the price are:

$$\frac{\partial^2 C}{\partial s_1^2} = \frac{A\sqrt{s_2}}{4s_1^{3/2}}\left(\frac{N'(d_1)}{\tilde{\gamma}_t}-N(d_1)\right) , \quad \frac{\partial^2 C}{\partial s_2^2} = \frac{A\sqrt{s_1}}{4s_2^{3/2}}\left(\frac{N'(d_1)}{\tilde{\gamma}_t}-N(d_1)\right) ,$$

$$\frac{\partial^2 C}{\partial s_1\partial s_2} = \frac{A}{4\sqrt{s_1s_2}}\left(\frac{N(d_1)}{\tilde{\gamma}_t}+N(d_1)\right) ,$$

because

$$\frac{\partial d_1}{\partial s_i} = \frac{1}{2\tilde{\gamma}_ts_i} , \quad i=1,2 .$$

We have that

$$\begin{aligned}A_{t,s} &= \bar{s}\cdot D^2C(t,s)\cdot\bar{s} = \frac{A\sqrt{s_1s_2}}{4}\begin{pmatrix}\frac{N'(d_1)}{\tilde{\gamma}_t}-N(d_1) & \frac{N'(d_1)}{\tilde{\gamma}_t}+N(d_1)\\ \frac{N'(d_1)}{\tilde{\gamma}_t}+N(d_1) & \frac{N'(d_1)}{\tilde{\gamma}_t}-N(d_1)\end{pmatrix} = \\ &= \frac{\sqrt{s_1s_2}}{4}\left(\frac{N'(d_1)}{\tilde{\gamma}_t}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}-N(d_1)\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\right) .\end{aligned}$$

We notice that this time the linear form is proportional to a linear combination of two constant matrices. In particular, we have that

$$A_{t,s} = \begin{pmatrix}\alpha-\beta & \alpha+\beta\\ \alpha+\beta & \alpha-\beta\end{pmatrix} ,$$

where

$$\alpha = \frac{\sqrt{s_1 s_2}}{4} \frac{N'(d_1)}{\tilde{\gamma}_t} > 0 \ , \quad \beta = \frac{\sqrt{s_1 s_2}}{4} N(d_1) > 0 \ .$$

If $\partial \Sigma$ is regular, then the solution $\bar{\gamma}$ of the maximization problem (11) is typically not constant, because it depends continuously on A, so on s_1, s_2 and t; but if $\partial \Sigma$ is not regular, it can happen that things change, as we shall see in the next section.

8. Stochastic correlation

Now we assume that

$$\Sigma = \left\{ \gamma \in \mathbb{R}^{2\times 2} \ \middle| \ \gamma = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}, \rho \in [-1,1] \right\} \ ,$$

where $\sigma_1 > \sigma_2 > 0$. This corresponds to the case when the volatilities σ_1, σ_2 of the two assets S_1 and S_2 are known, but the correlation is not. In particular we allow S_1 and S_2 to be in all the states between perfectly positively correlated ($\rho = 1$) and perfectly negatively correlated ($\rho = -1$), passing through a null correlation ($\rho = 0$). We indicate with S_1 the asset having higher volatility σ_1. Then we have that

$$\gamma\gamma^* = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \left(1-\rho^2\right)\sigma_2^2 \end{pmatrix} \ .$$

We represent $A_{t,s} = A$ as

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ .$$

Lemma 14. *If* $|b\sigma_1| > |c\sigma_2|$, *then:*

- *if* $b > 0$, *then the optimum is* $\bar{\rho} = 1$
- *if* $b < 0$, *then the optimum is* $\bar{\rho} = -1$

Proof. The function to maximize becomes:

$$f(\rho) = \mathrm{tr}\ (A\gamma\gamma^*) = a\sigma_1^2 + c\sigma_2^2 + 2b\rho\sigma_1\sigma_2 - c\sigma_2^2\rho^2 \ .$$

We have that, if $c \neq 0$, then f is a parabola, so

$$f'(\rho) = 2b\sigma_1\sigma_2 - 2c\sigma_2^2\rho \ .$$

The vertex of f is $\rho^* = \frac{b\sigma_1}{c\sigma_2}$; if $|\frac{b\sigma_1}{c\sigma_2}| > 1$, then $\rho^* \notin [-1,1]$, so the maximum is on the boundary of $[-1,1]$, that is $\bar{\rho} \in \{-1,1\}$. We suppose that $b > 0$; if $c > 0$, then $\rho^* > 1$, is the maximum of the parabola, so f is increasing in $[-1,1]$ and $\bar{\rho} = 1$; if $c < 0$, then $\rho^* < -1$ is the minimum of the parabola, so f is increasing in $[-1,1]$ and again $\bar{\rho} = 1$; at last if $c = 0$, f is a line with slope $2b\sigma_1\sigma_2 > 0$, so $\bar{\rho} = 1$. The case $b < 0$ is obtained with similar arguments. $\square$

Corollary 15 (Margrabe's exchange option). *If $C(t, s_1, s_2)$ is given by Eq. (16), then the optimal $\bar{\rho}$ is $\bar{\rho} = -1$, and the BSB equation becomes a BS equation with volatility*

$$\begin{pmatrix} \sigma_1 & 0 \\ -\sigma_2 & 0 \end{pmatrix} .$$

Proof. Since $b\sigma_1/(c\sigma_2) = -\sigma_1/\sigma_2 < -1$ and $b < 0$, we can apply Lemma 14 and Proposition 10. □

Corollary 16 (option on the geometrical mean). *If $C(t, s_1, s_2)$ is given by Eq. (18), then the optimal $\bar{\rho}$ is $\bar{\rho} = 1$, and the BSB equation becomes a BS equation with volatility*

$$\begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & 0 \end{pmatrix} .$$

Proof. We have $b = \alpha + \beta$ and $c = \alpha - \beta$. Since $\alpha, \beta > 0$, then $|\alpha + \beta| > |\alpha - \beta|$ and $\frac{\sigma_1}{\sigma_2} > 1$. Since $b = \alpha + \beta$, we may apply Lemma 14 and Proposition 10. □

In both cases we have seen that to apply Lemma 14 is necessary that $\sigma_1 > \sigma_2$. For this it is sufficient to denote the asset having higher volatility by S_1.

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