# Weak convergence of shortfall risk minimizing portfolios

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**Abstract**

In this paper we consider models of financial markets in discrete and continuos time case, and we show that under certain assumptions we can obtain tha weak convergence of various results about shortfall risk minimisation obtained so far in discrete time to similar ones in continuous time.

## 1 Introduction

We consider the paper [2] from which we will recall some results that we apply for the discrete time case and then find conditions in order to show the weak convergence of results obtained in discrete time case to results of continuous time case.

When modeling a financial market, the discounted price process of risky assets can be described as a semimartingale $S = (S(t))_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In this framework, let $H$ be a liability to be hedged at some future time, which can be put equal to 1

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without loss of generality. If $V(1)$ is the value at time 1 of a portfolio corresponding to a self-financing investment strategy, given by

$$V(t) = \tilde{V}_0 + \int_0^t \xi(u) dS(u) \quad \forall t \in [0,1],$$

where $\xi$ is a suitable process and $\tilde{V}_0$ is the initial capital, the *shortfall risk minimization problem* consists in determining the portfolio $V^*(1)$ which minimises the quantity

$$\mathbb{E}\left[l\left((H - V(1))^+\right)\right]$$

for a suitable "loss function" $l$, which is classically considered to be increasing, convex and continuously differentiable defined on $[0, \infty)$, and such that $\ell(0) = 0$. This problem is introduced for two reasons. First of all, if the market is incomplete it can be impossible to hedge perfectly $H$: in this case, a criterion has to be introduced to "minimise the risk" of having $H$ in one's portfolio in some sense, and the shortfall criterion has gained popularity in the last years. The second reason is that, whether the market is complete or incomplete, if one starts with an initial capital which is strictly less than the capital needed to (super)replicate the option, surely will end up at time 1 with a loss $H - V(1)$ which will be greater than zero with positive probability. The shortfall risk minimisation is then a measure of how much one is riskying by starting with an insufficient initial capital, even if the market is complete. We will also assume that

$$\mathbb{E}[l(H)] < \infty,$$

where $H := H(S(T))$ is a simple contingent claim of European type, where $H$ is a continuous function.

We will distinguish two cases: the case when the value of our final capital $V(1)$ is nonnegative, and the case when we do not impose constraints on $V$, apart from being admissible in a sense which we will specify.

**Definition 1.1.** *The shortfall risk is defined as the expectation*

$$\mathbb{E}\left[l\left((H - V(1))^+\right)\right]$$

*of the shortfall weighed by the loss function $l$.*

It can be found an admissible strategy which minimizes the shortfall risk while not using more capital than $\tilde{V}_0$. The optimization problem to be solved in this case is the following:

$$\begin{cases} \mathbb{E}[l\left((H - V(1)^+\right)] \to \min \\ V_0 \leq \tilde{V}_0 \end{cases}$$

In [2] the problem is reduced to the search of an element $\tilde{\phi}$ in the class

$$\mathcal{R} = \{\phi : \Omega \to [0,1] |\ \phi\ \mathcal{F}_T\text{-measurable}\}$$

which solves the following optimization problem:

$$\begin{cases} \min\limits_{\phi \in \mathcal{R}} \mathbb{E}[l((1-\phi)H)] \\ \sup\limits_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[\phi H] \leq \tilde{V}_0, \end{cases} \tag{1}$$

where $\mathcal{P}$ is the set of equivalent martingale measures.

### 1.1 Definition of weak convergence

For a given Polish space $E$ (that is a complete separable metric space) equipped with its Borel $\sigma$-field $\mathcal{E}$, consider the space $\mathcal{P}(E)$ of all probability measures on $(E, \mathcal{E})$. The set $\mathcal{P}(E)$ is endowed with the *weak topology* which is the coarsest topology for which the mapping $\mu \to \mu(f) = \int_E f d\mu$ is continuous for all bounded continuous functions $f$ on $E$. $\mathcal{P}(E)$ is itself a Polish space for this topology.

**Definition 1.2.** *The sequence $(\mu_n)_n$ of probability measures converges weakly to $\mu$ if, for every bounded continuous function $f$ on $E$ $(\mu_n(f))_n$ converges to $\mu(f)$.*

The weak convergence of random variables is defined from the weak convergence of the probability distributions. Let $X$ be an $E$-valued random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The image of $\mathbb{P}$ under $X$, denoted by $\mathbb{P}_X$ or equivalently by $\mathcal{L}(X)$ and belonging to $\mathcal{P}(E)$ is called the law or the distribution of $X$. Consider now a sequence $(X_n)_n$ of $E$-valued random variables, defined possibly on different spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$.

**Definition 1.3.** *$(X_n)_n$ converges in law if $(\mathcal{L}(X_n))_n$ converges weakly to $\mathcal{L}(X)$ in $\mathcal{P}(E)$. Notation: $X_n \overset{\mathcal{L}(E)}{\Rightarrow} X$.*

We will examine the case of rcll (right continuous left limited) processes which are $\mathbb{R}^d$-valued. Let $X$ be such a process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $X$ may be considered as a random variable taking its values in the Polish space $E = \mathbb{D}(\mathbb{R}^d)$ (also called Shorokhod space) equipped with the Skorokhod topology (see [4, Theorem 1.2.2,p.66]). Thus the distribution $\mathcal{L}(X)$ is an element of $\mathcal{P}(\mathbb{D}(\mathbb{R}^d))$.

We will apply this general framework to two cases. First to a sequence of discrete time models, which we will extend in continuous time by assuming that the price paths will be pathwise constant. Then, to continuous time models driven by Brownian motion, with prices having continuous sample paths. The final scope is to prove that, if a sequence of discrete time models converge weakly to a continuous time model driven by a Brownian motion, then also the optimal portfolios for the shortfall risk converge weakly.

## 2 Explicit solutions for the shortfall risk minimising strategies in the complete case

In the following we suppose that the set of equivalent martingale measures contains a unique element, so the market is complete, and we denote by $\rho^*(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$ the corresponding Radon-Nikodym derivative process. Since we are more interested in the values of the process $\rho^*$ at time 1, we will consider the density $\rho^*(1)$

Now we distinguish between two cases: the case when we impose that the self-financing portfolio of the agent remains positive, and the case when we do not impose constraints on the portfolio. In both these situations, the shortfall risk minimisation problem has a solution both in discrete as in continuous time.

We start from the constrained case. The next result, taken from [2], is valid for a general complete market.

**Proposition 2.1.** *The solution $\tilde{\phi} \in \mathcal{R}$ for Problem (1) under the constraint $0 \leq V_T \leq H$ is given by*

$$\tilde{\phi} = \begin{cases} 1 - \left(\frac{I(c\rho^*(1))}{H} \wedge 1\right) & on \ \{H > 0\} \\ 1 & on \ \{H = 0\} \end{cases}$$

*The constant $c$ is determined by the condition*

$$\mathbb{E}^*[\tilde{\phi} H] = \tilde{V}_0,$$

Now we pass to the unconstrained case. For this, we consider a result in [1], which gives us a characterization of the optimal strategies in the particular case when the loss function is convex.

**Definition 2.1.** *We define the set of the modified contingent claims as*

$$\mathcal{X} := \left\{ X | \ X \leq H(a.s), \ \mathbb{E}^*[X] \leq \tilde{V}_0 \right\}$$

$\mathcal{X}$ is the set of all claims less than $H$ which can be replicated with initial capital (less than or equal to) $V_0$. Consider now the shortfall risk minimization problem:

$$\min_{X\in\mathcal{X}} \mathbb{E}[l(H(S_N)-X)]. \tag{2}$$

The modified contingent claim that solves Problem (2) coincides with the payoff of the optimal portfolio for the shortfall risk minimization problem. In the following we will maintain the assumptions from the previous subsection made for the function $I := (l')^{-1}$ and also the assumptions $A1 - A5$.

**Theorem 2.1.** *Define the modified contingent claim*

$$X^* := H - I\left(c^*\rho^*(1)\right)$$

*with $c^* > 0$ such that $\mathbb{E}^*[X^*] = \tilde{V}_0$. Then $X^*$ solves Problem (2).*

Considering these results, we start dealing with the discrete time case. We will apply the previous proposition to a particular sequence of discrete time models.

## 2.1 Preliminaries for dicrete time case

In this section we will state the hypothesis in which we will work:

- A1 Consider the probability spaces as being of the form $\Omega_n = \{0,1\}^n$; we denote by $\omega = (\omega_1, \ldots \omega_n)$ an element of $\Omega_n$ and by $\overline{\omega}_k = (\omega_1, \ldots \omega_k)$ the first $k$ positions of $\omega$;
- A2 $\mathcal{F}_{n,k} = \left\{A_{n,k} \times \{0,1\}^{n-k} | A_{n,k} \subset \{0,1\}^k\right\}$ is the associated families of $\sigma-$algebras.

Since we suppose we have a unique martingale measure then we know that there are no arbitrage opportunities for our multiperiod model and so it is possible to construct one-period conditional probabilities that are compatible with the risk neutrality. The martingale measure $\mathbb{Q}$ can be computed from these conditional probabilities by multiplying them together in accordance with the information structure of the multiperiod model.

¿From [3, Section 3.4, pp. 96] we have the following result:

**Lemma 2.1.** *If the multiperiod model does not have any arbitrage opportunities, then none of the underlying single period models has any arbitrage opportunities in the single period sense.*

At this point we state the following:

A3 $\forall k, \overline{\omega}_{k-1}, \omega_k$, we denote by $\mathbb{P}_{n,k}^{\overline{\omega}_{k-1}}(\omega_k) > 0$ the probabilities conditioned to $\overline{\omega}_{k-1}$ of the intermediate single period markets so $\mathbb{P}_{n,k}(\omega_k$ will denote the unconditional probability for the corresponding markets and by $\mathbb{P}_n(\omega) = \prod_{k=1}^{n} \mathbb{P}_{n,k}^{\overline{\omega}_{k-1}}(\omega_k)$ the probability measure corresponding to the space $\Omega_n$;

Now consider the probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ and consider a partition $\{t_1, \dots, t_n\}$ for the time interval $[0,1]$. The price processes $(B_{n,k}; S_{n,k})_k$ are defined in discrete time in the following way: $B_{n,k}$ denotes the riskless bond price at time $t_k$ and $S_{n,k}$ denotes the stock prices at $t_k$. Consider the following dynamic for the price process:

$$\begin{cases} B_{n,k} = B_{n,k-1}(1 + Y_{n,k}^B) \\ S_{n,k} = S_{n,k-1}(1 + Y_{n,k}^S) \end{cases}$$

where $Y_{n,k}^B, Y_{n,k}^S$ denote the returns of the corresponding assets.

A4 $\forall k, \overline{\omega}_{k-1}, Y_{n,k}^S(\overline{\omega}_{k-1}, 0) < Y_{n,k}^B < Y_{n,k}^S(\overline{\omega}_{k-1}, 1)$ where $\left(Y_{n,k}^S\right)$ is a sequence of independently identically distributed random variables such that

$$\forall k, \overline{\omega}_{k-1}, \omega_k,\ \mathbb{P}_{n,k}^{\overline{\omega}_{k-1}}(\omega_k) = \mathbb{P}_{n,k}(\omega_k) = \frac{1}{2}.$$

A5 We define:

$$\begin{aligned} Y_{n,k}^B &= \frac{1}{n}\alpha^B\left(\frac{k}{n}\right) \\ Y_{n,k}^S &= \frac{1}{n}\alpha^S\left(\frac{k}{n}\right) + \beta^S\left(\frac{k}{n}\right)\Delta_{n,k} \end{aligned}$$

where $\Delta_{n,k}$ is a sequence of i.i.d. random variables defined as:

$$\Delta_{n,k} = \begin{cases} \frac{-1}{\sqrt{n}} & \text{if } \omega_k = 0, \\ \frac{1}{\sqrt{n}} & \text{if } \omega_k = 1 \end{cases}$$

so that:

$$\mathbb{P}_n\left[\Delta_{n,k} = -\frac{1}{\sqrt{n}}\right] = \mathbb{P}_n\left[\Delta_{n,k} = \frac{1}{\sqrt{n}}\right] = \frac{1}{2}$$

and $\alpha^B, \alpha^S, \beta^S$ are nonnegative continuous functions on $[0,1]$, $\alpha^S > \alpha^B$.

### 2.2 Preliminaries for the continuous time case

The model for the continuous time financial market that we will use is the following:

$$\begin{cases} B(t) = B_0 \exp\left\{\int_0^t \alpha^B(u)du\right\} \\ S(t) = S_0 \exp\left\{\int_0^t \left(\alpha^S(u) - \frac{1}{2}\left(\beta^S\right)^2(u)\right)du + \int_0^t \beta^S(u)dW(u)\right\}, \end{cases} \quad t < 1.$$

We have chosen this model because it represents the limit under the weak convergence of the discrete time models built in the previous section. The main tool in our work is the joint convergence of the sequences of stock prices and Radon-Nykodim processes. How do we pass from the discrete time model to this continuous time model? We have considered the probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)_n$ and a time interval equal to $[0,1]$. For this interval we chose some intermediate time points $t_k$ for which we will have the corresponding price processes $(B_{n,k}; S_{n,k})_k$. Now in order to be able to pass to the limit we will make the following notation:

$$\overline{S}_n(t) := S_{n,[nt]}$$

same kind of notation being valid for all processes in discrete time.

# 3 Results of weak convergence

### 3.1 Optimization problem with constraints

¿From Proposition 2.1 we obtain a sequence $(\phi_n)_n$ of solutions for each of the optimization problems in discrete time and a sequence of constants $(c_n)_n$ each of which is determined by the condition $\mathbb{E}_n^*[\phi_n H] = \tilde{V}_0$.

**Proposition 3.1.** *Under the assumptions A1-A5 and that $\exists \alpha < 0$, $\exists c_0 > 0$ s.t. $I(x) \leq c_0 x^\alpha$ for all $x \in \mathbb{R}^+$ we have that $c_n \to c$.*

*Proof.* Define

$$\begin{aligned} \Delta_n &:= \mathbb{E}_n^*[H(\overline{S}_n(1))] - \tilde{V}_0 = \mathbb{E}_n^*\left[I(c_n \overline{\rho}_n^*(1)) \wedge H(\overline{S}_n(1))\right] \\ \Delta &:= \mathbb{E}^*[H(S(1))] - \tilde{V}_0 = \mathbb{E}^*\left[I(c\rho(1)) \wedge H(S(1))\right] = \mathbb{E}^*[S(1)] - \tilde{V}_0 \end{aligned}$$

¿From [4, Section 2.1, pp. 169,178] we have that:

$$(\overline{\rho}_n^*, \overline{S}_n) \overset{\mathcal{L}(\mathbb{D}^1)}{\Rightarrow} (\rho^*, S).$$

For a given $x > 0$, denote by $f : \mathbb{D}(\mathbb{R}^2) \to \mathbb{R}$ the function defined as

$$f(\overline{\rho}_n^*, \overline{S}_n) := I(x\overline{\rho}_n^*(1)) \wedge H(\overline{S}_n(1)).$$

Then $f$ is continuous, since is is the minimum of two continuous functions. ¿From [4, p. 172] we have that $(\rho_n^\alpha)_n$ is uniformly integrable and so $\{I(x\overline{\rho}_n^*(1))\}_n$ is uniformly integrable. But $f(\overline{\rho}_n^*, \overline{S}_n) \leq I(x_n\rho_n^*(1))$ for all $n$. hence we obtain also that $\{f(\overline{\rho}_n^*, \overline{S}_n)\}_n$ is uniformly integrable.

Consider now some bounded and continuous function $g : \mathbb{R}^{d+1} \to \mathbb{R}$. Then $g \circ f$ is a bounded continuous function and together with the weak convergence of the pair $(\overline{\rho}_n^*, \overline{S}_n)$ we have the following convergence

$$\mathbb{E}_n^* \left[g \circ f(\overline{\rho}_n^*, \overline{S}_n)\right] \to \mathbb{E}^* \left[g \circ f(\rho^*, S)\right]$$

which written in another way yields:

$$\mathbb{E}_n^* \left[g\left(f(\overline{\rho}_n^*, \overline{S}_n)\right)\right] \to \mathbb{E}^* \left[g(f(\rho^*, S))\right]$$

and remembering that this holds for all bounded continuous function $g$ we have that:

$$f(\overline{\rho}_n^*, \overline{S}_n) \overset{\mathcal{L}(\mathbb{R})}{\Rightarrow} f(\rho^*, S).$$

Since $f(\overline{\rho}_n^*, \overline{S}_n)_n$ is uniformly integrable we obtain:

$$\mathbb{E}_n^* \left[I(x\overline{\rho}_n^*(1)) \wedge H(\overline{S}_n(1))\right] \to \mathbb{E}^* \left[I(x\rho^*(1)) \wedge H(S(1))\right]$$

Now the functions $\phi_n$ that associate $\mathbb{E}_n^* \left[I(x\overline{\rho}_n^*) \wedge H(\overline{S}_n)\right]$ to $x$ are continuous, strictly increasing and converge pointwise to the function $\phi(\cdot) = \mathbb{E}^* \left[I(\cdot\rho^*(1)) \wedge H(S(1))\right]$. Then also $\phi_n^{-1} \to \phi^{-1}$ pointwise. The functions $\phi_n^{-1}, \phi^{-1}$ are also continuous so by applying Dini's theorem we get that $\phi_n^{-1}$ converge uniformly on compact sets to $\phi^{-1}$. But $\Delta_n, \Delta$ belong to a compact set in $\mathbb{R}^+$ since from the convergence of the expected values of the contingent claims we have that $\Delta_n \to \Delta$, so:

$$c_n = \phi_n^{-1}(\Delta_n) \to \phi^{-1}(\Delta) = c \Longrightarrow c_n \to c.$$

□

**Proposition 3.2.** *Under the same assumptions as in Proposition 3.1,* $\tilde{\phi}_n H \overset{\mathcal{L}(\mathbb{R})}{\Rightarrow} \tilde{\phi} H$*, where* $\tilde{\phi}$ *is defined as in Proposition 3.1.*

*Proof.* Recall that

$$\begin{aligned}\tilde{\phi}_n &= 1-\left(\frac{I(c_n\rho_n^*(1))}{H}\wedge 1\right) \quad \text{on } \{H>0\}\\ \tilde{\phi} &= 1-\left(\frac{I(c\rho^*(1))}{H}\wedge 1\right) \quad \text{on } \{H=0\}\end{aligned}$$

Consider the following notation $f(\overline{\rho}_n^*, \overline{S}_n) := H(\overline{S}_n(1)) - \big(I(x\overline{\rho}_n^*) \wedge H(\overline{S}_n(1))\big)$ so $f : \mathbb{D}(\mathbb{R}^2) \to \mathbb{R}$ is a continuous function. Then for all bounded continuous function $g$ defined on $\mathbb{R}^2$, $g \circ f$ is a bounded continuous function. This together with the weak convergence of the pair $(\overline{\rho}_n^*, \overline{S}_n)$ yield the following convergence:

$$\begin{aligned}&\mathbb{E}_n^*\left[g\left(H(\overline{S}_n(1)) - \big(I(x\overline{\rho}_n^*(1)) \wedge H(\overline{S}_n(1))\big)\right)\right]\\ &\quad \to \mathbb{E}^*\left[g\left(H(S(1)) - (I(x\rho^*(1)) \wedge H(S(1)))\right)\right]\end{aligned}$$

The functions $\psi_n$ that associate $\mathbb{E}_n^*\left[g\left(H(\overline{S}_n(1)) - \big(I(x\overline{\rho}_n^*) \wedge H(\overline{S}_n(1))\big)\right)\right]$ to $x$ are continuous and converge pointwise to the function

$$\psi(\cdot) := \mathbb{E}^*\left[g\left(H(S(1)) - (I(\cdot\rho^*) \wedge H(S(1)))\right)\right]$$

which is also continuous. By Dini's theorem we have that $\phi_n$ converge uniformly to $\psi$ on compact sets.

At this point we use Proposition 3.1 that tells us that $c_n \to c$. So $c_n, c$ belong to some compact set in $\mathbb{R}^+$. But this means that $\psi_n(c_n) \to \psi(c)$. Recalling the definitions of $\psi_n, \psi$ this can be written in the following way:

$$\begin{aligned}&\mathbb{E}_n^*\left[g\left(H(\overline{S}_n(1)) - \big(I(c_n\overline{\rho}_n^*(1)) \wedge H(\overline{S}_n(1))\big)\right)\right]\\ &\quad \to \mathbb{E}^*\left[g\left(H(S(1)) - (I(c\rho^*(1)) \wedge H(S(1)))\right)\right]\end{aligned}$$

But this relation holds for all bounded continuous functions $g$ and so we obtain the weak convergence of the modified claims $\tilde{\phi}_n H$:

$$\begin{aligned}&\tilde{\phi}_n H(\overline{S}_n(1)) = (H(\overline{S}_n(1)) - \big(I(c_n\overline{\rho}_n^*) \wedge H(\overline{S}_n(1))\big)\\ &\overset{\mathcal{L}(\mathbb{R})}{\Longrightarrow} H(S(1)) - (I(c\rho^*) \wedge H(S(1))) = \tilde{\phi} H(S(1))\end{aligned}$$

□

### 3.2 Optimization problem without constraints

**Proposition 3.3.** *Under the assumptions of Proposition 3.1 we have the weak convergence of the modified contingent claims given by Theorem 2.1.*

*Proof.* Recall that

$$\begin{aligned} X_n^* &= H(\overline{S}_n(1)) - I(c_n^* \overline{\rho}_n^*(1)) \\ X^* &= H(S(1)) - I(c^* \rho_n(1)) \end{aligned}$$

when $c_{(n)}^*$ is given by $\mathbb{E}_{(n)}^*[X_{(n)}^*] = \tilde{V}_0$. Define now

$$\begin{aligned} \Delta_n &:= = \mathbb{E}_n^*[H(\overline{S}_n(1))] - \tilde{V}_0 = \mathbb{E}_n^*\left[I\left(c_n^* \overline{\rho}_n^*(1)\right)\right] \\ \Delta &:= \mathbb{E}^*[H(S(1))] - \tilde{V}_0 = \mathbb{E}^*[I(c^* \rho^*(1))] \end{aligned}$$

In the first part of our proof we show that $c_n^* \to c^*$. If we consider the functions $\phi_n(x) := I\left(c_n^* \overline{\rho}_n^*(1)\right)$ and $\phi := I(c^* \rho^*(1))$ then by the same arguments used in the proof of Proposition 3.1 we obtain that $\phi_n \stackrel{\mathcal{L}(\mathbb{R})}{\Rightarrow} \phi$ which together with the assumption on $I$ that yields the uniform integrability of $\phi_n,\ \phi$, leads to the convergence:

$$\mathbb{E}_n^*\left[I\left(x\overline{\rho}_n^*(1)\right)\right] \to \mathbb{E}^*[I(x\rho^*(1))].$$

The functions

$$\begin{aligned} \psi_n(x) &:= \mathbb{E}_n^*[I(x\overline{\rho}_n^*(1))] \\ \psi(x) &:= \mathbb{E}^*[I(x\rho^*(1))] \end{aligned}$$

are continuous and increasing. Then also the inverse functions are continuous and convergent so that by Dini's theorem we obtain that this functions converge uniformly on compact sets. But $\Delta_n \to \Delta$ since we have the convergence of the underlying assets and $H$ is continuous and this yields the convergence

$$c_n^* = \psi_n^{-1}(\Delta_n) \to \psi^{-1} = c^*.$$

Now we proceed in proving the convergence of the modified claims. $X_n^*, X^*$ are continuous functions of two variables. Since we have the joint convergence $(\overline{\rho}_n^*, \overline{S}_n) \stackrel{\mathcal{L}(\mathbb{D}^{d+1})}{\Rightarrow} (\rho^*, S)$ we obtain that

$$\phi_n(\overline{\rho}_n^*, \overline{S}_n) := H(\overline{S}_n(1)) - I\left(x\overline{\rho}_n^*(1)\right) \stackrel{\mathcal{L}(\mathbb{R})}{\Rightarrow} H(S(1)) - I(x\rho^*) = \phi(\rho^*, S)$$

since for all bounded continuous functions $g$ we have that $\mathbb{E}_n^*[g(\phi_n)] \to \mathbb{E}^*[g(\phi)]$. Now the functions

$$f_n(x) = \mathbb{E}_n^*[g(H(\overline{S}_n(1)) - I(x\overline{\rho}_n^*(1)))]$$

are continuous and convergent to the function $f(x) = \mathbb{E}^*[g(H(S(1)) - I(x\rho^*(1)))]$. Since we have that $c_n^* \to c^*$, then $f(c_n^*) \to f(c^*)$ which means that

$$X_n^* = H(\overline{S}_n(1)) - I(c_n^*\overline{\rho}_n^*(1)) \overset{\mathcal{L}(\mathbb{R})}{\Longrightarrow} H(S(1)) - I(c^*\rho^*) = X^*$$

which is what we wanted to prove. $\square$

# 4 Final remarks

For the discrete time model for which we showed the weak convergence of the modified claims we can take $S_{n,k}$ to be a $d$-dimensional system of stock prices at time $t_k$ and consider a more general type of returns. That will mantain the weak convergence of the pair:

$$(\overline{\rho}_n^*, \overline{S}_n) \overset{\mathcal{L}(\mathbb{D}^{d+1})}{\Rightarrow} (\rho^*, S).$$

Consider for example the following construction of returns:

$$\begin{cases} Y_{n,k}^B = \frac{1}{n}\alpha^B\left(\frac{k}{n}\right) \\ Y_{n,k}^S = \frac{1}{n}\alpha^S\left(S_{n,k-1}\right) + \beta^S\left(S_{n,k-1}\right)\Delta_{n,k} \end{cases}$$

where:

- for each value $x \in \mathbb{R}^d$, $\beta^S(x)$ is a $(d+1) \times (d+1)$ invertible matrix;
- $\Delta_{n,k} = (\Delta_{n,k}^{(1)}, \ldots, \Delta_{n,k}^{(d)})'$ is a random vector satisfying

  $$\mathbb{E}[\Delta_{n,k}] = 0 \text{ and } \mathbb{E}[\Delta_{n,k}\Delta_{n,k}'] = I_d$$

  where $I_d$ is the unitary matrix.

To get uncorrelated $(\Delta_{n,k}^{(i)})_{i \leq d}$ and a discrete complete market, we can construct $d$ random variables taking $d+1$ values. Then considering the construction presended in [4, pp. 176-178] we have:

$$(\overline{\rho}_n^*, \overline{S}_n) \overset{\mathcal{L}(\mathbb{D}^1)}{\Rightarrow} (\rho^*, S).$$

For proof see [4, pp. 178].

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