Appendix A

Positivity of rates

One of the main weaknesses of Gaussian interest rates models is the possibility of negative rates with positive probability. This appendix partially investigates this point. First of all we concentrate on the one-dimensional Vasicek model, and calculate the probability that there will be a negative forward rate at a certain time; this probability being 1, we calculate the probability that there will be a negative forward rate before a fixed instant T. Later, we do all this in the more general finite dimensional Gaussian model.

A.1 The Vasicek model

We take $R \geq 0$, and we define

$$\tau_R = \inf\{t \mid r_t < 0 \text{ or } r_t > R\}, \quad \tau = \inf\{t \mid r_t < 0\}$$

We notice that $\tau_R \to \tau$ a.s. as R tends to $+\infty$.

Lemma A.1 $\tau < +\infty$ *a.s.*

Proof. First of all we search for a function h solution of $Ah = 0$, where A is the second order operator associated to $(r_t)_t$, that is:

$$Ah(x) = a(b-x)h'(x) + \frac{1}{2}\sigma^2 h''(x) \tag{A.1}$$

Lemma A.2 *A solution of $Ah = 0$ is of the form*

$$h(r) = D + C\int_0^r e^{\frac{a}{\sigma^2}(y-b)^2}\, dy$$

where C, D are arbitrary real numbers.

Proof. We define $g = h'$; then the equation $Ah = 0$ becomes:

$$g'(x) = \frac{2a(x-b)}{\sigma^2} g(x)$$

then

$$g(x) = Ce^{\int \frac{2a(y-b)}{\sigma^2}\, dy} = Ce^{\frac{a(x-b)^2}{\sigma^2}}$$

since $h = \int v(x)\, dx$, we have the thesis. $\diamond$

If h is of the above form, then $M_t = h(r_{t\wedge\tau_R})$ is a bounded martingale (because $0 \le r_{t\wedge\tau_R} \le R$ and h is continuous), so it converges a.s.; but, if $\tau_R = +\infty$, then $h(r_{t\wedge\tau_R}) = h(r_t)$ cannot converge, so $\tau_r < +\infty$ a.s.; besides, since $(M_t)_t$ is a bounded martingale, we have $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ $\forall t \ge 0$, so:

$$h(x) = \mathbb{E}[h(r_0)] = \lim_{t\to+\infty} \mathbb{E}[h(r_{t\wedge\tau_R})] = \mathbb{E}\left[\lim_{t\to+\infty} h(r_{t\wedge\tau_R})\right] = \mathbb{E}[h(r_{\tau_R})]$$

where the third equality follows from Lebesgue theorem, so:

$$h(x) = h(0)\mathbb{P}\{r_{\tau_R} = 0\} + h(R)\mathbb{P}\{r_{\tau_R} = R\} = h(0)\mathbb{P}\{r_{\tau_R} = 0\} + h(R)(1 - \mathbb{P}\{r_{\tau_R} = 0\})$$

and we obtain:

$$\mathbb{P}\{r_{\tau_R} = 0\} = \frac{h(x) - h(R)}{h(0) - h(R)} = \frac{\displaystyle\int_x^R e^{\frac{a}{\sigma^2}(y-b)^2}\, dy}{\displaystyle\int_0^R e^{\frac{a}{\sigma^2}(y-b)^2}\, dy}$$

We call $v(x,R) = \mathbb{P}\{r_{\tau_R} = 0\}$ under the condition $r_0 = x$. Then we have the intuitive conditions $v(0,R) = 1$ and $v(R,R) = 0$. Moreover we have that

$$\{\tau < +\infty\} = \{\exists R \text{ such that } r_{\tau_R} = 0\} = \bigcup_{R>0} \{r_{\tau_R} = 0\}$$

so

$$\mathbb{P}\{\tau < +\infty\} = \lim_{R\to+\infty} \{r_{\tau_R} = 0\} = \lim_{R\to+\infty} v(x,R) = 1$$

$\diamond$

We have seen that $\mathbb{P}\{\tau < +\infty\} = 1$, that is there exists almost surely a T such that $r_T < 0$. Now, let's fix $T > 0$, and calculate $\mathbb{P}\{\tau \le T\}$, that is the probability that the spot rate is positive up to the time T. First of all we give the result in terms of the solution of a parabolic PDE with boundary condition.

Lemma A.3 *If there exist a solution $v(t,x) \in C_b^1$ of the parabolic partial differential equation*

$$\begin{cases} \dfrac{\partial v}{\partial t} + Av = 0 & (t,x) \in [0,T) \times (0,+\infty) \\ \\ v(T,x) = 0 & x \in (0,+\infty) \\ v(t,0) = 1 & t \in [0,T) \end{cases}$$

where A is given by Eq. (A.1), then $\mathbb{P}\{\tau \le T\} = v(0,r_0)$.

Proof. We apply Itô formula to the function v:

$$dv(t, r_t) = \left(\frac{\partial v}{\partial t} + Av\right)(t, r_t)\ dt + \frac{\partial v}{\partial x}(t, r_t)\sigma\ dW_t$$

then, because of the PDE, we have:

$$v(T \wedge \tau, r_{T\wedge\tau}) = v(0, r_0) + \int_0^{T\wedge\tau} \frac{\partial v}{\partial x}(t, r_t)\sigma\ dW_t$$

making the expectation, since $v \in C_b^1$, then the expectation of the stochastic integral is 0, and we get:

$$\begin{aligned} v(0, r_0) &= \mathbb{E}[v(T \wedge \tau, r_{T\wedge\tau})] = \mathbb{E}[v(T, r_T)\mathbf{1}_{\tau>T} + v(\tau, r_\tau)\mathbf{1}_{\tau\leq T}] = \\ &= \mathbb{E}[0 \cdot \mathbf{1}_{\tau>T} + 1 \cdot \mathbf{1}_{\tau\leq T}] = \mathbb{P}\{\tau \leq T\} \end{aligned}$$

$\diamond$

The problem with this approach is that there is not an explicit solution to this PDE, so we have no idea of the behaviour of the solution. The following proposition partially answers this question, giving an estimation from above to the searched probability.

Proposition A.4

$$\mathbb{P}\{\tau \leq T\} \leq \frac{\sigma\sqrt{e^{2aT} - 1}}{2x\sqrt{\pi a}}$$

Proof. We notice that

$$\begin{aligned} \mathbb{P}\{\tau \leq T\} &= \mathbb{P}\left\{\min_{t\in[0,T]} r_t \leq 0\right\} = \mathbb{P}\left\{\max_{t\in[0,T]} (-r_t) \geq 0\right\} = \mathbb{P}\left\{\max_{t\in[0,T]} (-e^{at} r_t) \geq 0\right\} = \\ &= \mathbb{P}\left\{\max_{t\in[0,T]} \left(-b(e^{at} - 1) - \int_0^t e^{as}\sigma\ dW_s\right) \geq r_0\right\} \leq \\ &\leq \mathbb{P}\left\{\max_{t\in[0,T]} \left(-\int_0^t e^{as}\sigma\ dW_s\right) \geq r_0\right\} \end{aligned}$$

the process in the last expression is a martingale, so we can apply Kolmogorov's maximal inequality, and we get:

$$\mathbb{P}\{\tau \leq T\} \leq \frac{1}{r_0}\mathbb{E}\left[\left(\int_0^T e^{as}\ dW_s\right)^+\right]$$

Now we need the

Lemma A.5 *If X is a random variable with law $N(0,1)$, then* $\mathbb{E}[X^+] = (2\pi)^{-1/2}$.

Proof.

$$E[X^+] = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x e^{-\frac{1}{2}x^2}\, dx = \frac{1}{\sqrt{2\pi}} [-e^{-\frac{1}{2}x^2}]_0^{+\infty} = \frac{1}{\sqrt{2\pi}}$$

$\diamond$

Since $\int_0^T e^{as}\, dW_s$ has law $N(0, \int_0^T e^{2as}\, ds)$, that is $N(0, \frac{\sigma^2}{2a}(e^{2aT}-1))$, then we can calculate the last expectation, and finally we get:

$$\mathbb{P}\{\tau \leq T\} \leq \frac{1}{r_0}\sqrt{\frac{\sigma^2}{2a}(e^{2aT}-1)} = \frac{\sigma\sqrt{e^{2aT}-1}}{2r_0\sqrt{\pi a}}$$

$\diamond$

Now we try a different approach. We see the stochastic integral in the solution as a Brownian motion with respect to another time scale, that is the one of the local times.

We define:

$$c(t) = \int_0^t (e^{as}\sigma)^2\, ds = \frac{\sigma^2}{2a}(e^{2at}-1)$$

and the **local time**:

$$\tau(u) = c^{-1}(u) = \frac{1}{2a}\log\left(1 + \frac{2a}{\sigma^2}u\right)$$

then $\tau(u)$ are (deterministic) stopping times such that

$$B(u) = \int_0^{\tau(u)} e^{as}\sigma\, dW_s$$

is a Brownian motion with respect to the filtration $(\mathcal{F}_{\tau(u)})_t$; then we can write:

$$b(e^{a\tau(u)}-1) = b\left(\exp\left(a\frac{1}{2a}\log\left(1+\frac{2a}{\sigma^2}u\right)\right)-1\right) = b\left(\sqrt{1+\frac{2a}{\sigma^2}u}-1\right)$$

and we finally obtain:

$$e^{a\tau(u)}r_{\tau(u)} = r_0 + b\left(\sqrt{1+\frac{2a}{\sigma^2}u}-1\right) + B(u), \quad 0 \leq u \leq c(T) = \frac{\sigma^2}{2a}(e^{2at}-1)$$

Now we define:

$$\begin{aligned} X_u &= r_0 + b\left(\sqrt{1+\frac{2a}{\sigma^2}u}-1\right) + B(u) = r_0 + \phi(u) + B(u), \\ Y_u &= r_0 + B(u) \end{aligned}$$

then the processes X and Y induce the laws $\mathbb{P}_X$ and $\mathbb{P}_Y$ on $C_0([0, c(T)])$, which are equivalent and which mutual density is

$$\begin{aligned}
\frac{d\mathbb{P}_X}{d\mathbb{P}_Y} &= \exp\left(\int_0^{c(T)} \phi'(t)\ dY_t - \frac{1}{2}\int_0^{c(T)} \phi'(t)^2\ dt\right) = \\
&= \exp\left(\int_0^{c(T)} \phi'(t)\ dB(t) - \frac{1}{2}\int_0^{c(T)} \phi'(t)^2\ dt\right)
\end{aligned}$$

Applying Hölder's inequality, we have the following extimation:

$$\begin{aligned}
\mathbb{P}\{\tau \leq T\} &= \mathbb{P}\left\{\min_{u\in[0,c(T)]} X_u \leq 0\right\} = \int_\Omega \mathbf{1}_{\min_{u\in[0,c(T)]} X_u \leq 0}\ d\mathbb{P}_X = \\
&= \int_\Omega \mathbf{1}_{\min_{u\in[0,c(T)]} Y_u \leq 0} \frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\ d\mathbb{P}_Y \leq \mathbb{E}[\mathbf{1}^p_{\min_{u\in[0,c(T)]} Y_u \leq 0}]^{1/p} \mathbb{E}\left[\left(\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\right)^q\right]^{1/q} = \\
&= \left(\mathbb{P}\left\{\min_{u\in[0,c(T)]} Y_u \leq 0\right\}\right)^{1/p} \mathbb{E}\left[\left(\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\right)^q\right]^{1/q}
\end{aligned}$$

for all $p, q > 1$ such that $1/p + 1/q = 1$. We have that:

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\right)^q\right] &= \mathbb{E}\left[\exp\left(q\int_0^{c(T)} \phi'(t)\ dB(t) - \frac{q}{2}\int_0^{c(T)} \phi'(t)^2\ dt\right)\right] = \\
&= \exp\left(-\frac{q}{2}\int_0^{c(T)} \phi'(t)^2\ dt\right) \mathbb{E}\left[\exp\left(q\int_0^{c(T)} \phi'(t)\ dB(t)\right)\right] = \\
&= \exp\left(\left(\frac{q^2}{2} - \frac{q}{2}\right)\int_0^{c(T)} \phi'(t)^2\ dt\right)
\end{aligned}$$

and:

$$\begin{aligned}
\int_0^{c(T)} \phi'(t)^2\ dt &= \int_0^{c(T)} \frac{b^2 a^2\ du}{\sigma^4(1 + \frac{2a}{\sigma^2}u)} = \int_0^{c(T)} \frac{b^2\ du}{\frac{2\sigma^2}{a}(\frac{\sigma^2}{2a} + u)} = \\
&= \left[\frac{b^2 a}{2\sigma^2}\log\left(u + \frac{\sigma^2}{2a}\right)\right]_0^{c(T)} = \frac{b^2 a}{2\sigma^2}\log\left(\frac{\frac{\sigma^2}{2a}e^{2aT}}{\frac{\sigma^2}{2a}}\right) = \frac{b^2 a^2 T}{\sigma^2}
\end{aligned}$$

Moreover, if we call $\tau_a = \inf\{t \mid B(t) = a\}$ the stopping time of the first passage of the Brownian motion from a, then

$$\begin{aligned}
\mathbb{P}\left\{\min_{u\in[0,c(T)]} Y_u \leq 0\right\} &= \mathbb{P}\left\{\min_{u\in[0,c(T)]} B(u) \leq -r_0\right\} = \mathbb{P}\{\tau_{-r_0} \leq c(T)\} = \\
&= \frac{2}{\sqrt{2\pi c(T)}}\int_{-r_0}^{+\infty} e^{-\frac{x^2}{2c(T)}}\ dx = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\frac{r_0}{c(T)}} e^{-\frac{y^2}{2}}\ dy
\end{aligned}$$

then we arrive to:

$$\begin{aligned}\mathbb{P}\{\tau \le T\} &= \left(2N\left(\frac{r_0}{c(T)}\right)\right)^{1/p} \exp\left((q-1)\frac{b^2a^2T}{2\sigma^2}\right) = \\ &= \left(2N\left(\frac{r_0}{c(T)}\right)\right)^{1/p} \exp\left(\frac{1}{p-1}\frac{b^2a^2T}{2\sigma^2}\right)\end{aligned}$$

for all $p > 1$, where N is the distribution function of the normal law $N(0,1)$:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}\, dx$$

We can try to find the optimal exponent p_1, that is the exponent $p_1 > 1$ such that the quantity on the right hand side is minimal.

Lemma A.6 *The optimal exponent is*

$$p_1 = \frac{\log A - \sqrt{-B\log A}}{B + \log A}$$

where

$$A = 2N\left(\frac{r_0}{c(T)}\right), \quad B = \frac{b^2a^2T}{2\sigma^2}$$

Proof. We have to minimize the function

$$f(p) = A^{1/p}\exp\left(\frac{B}{p-1}\right) = \exp\left(\frac{\log A}{p} + \frac{B}{p-1}\right)$$

We notice that minimizing f is equivalent to minimize its logarithm, then we calculate $(\log f)'$:

$$\frac{d}{dp}\left(\frac{\log A}{p} + \frac{B}{p-1}\right) = -\frac{\log A}{p^2} - \frac{B}{(p-1)^2}$$

Imposing $(\log f)' \ge 0$, we find:

$$p^2(\log A + B) - 2p\log A + \log A \le 0$$

$(\log f)'$ is decreasing from 1 to p_1, then it is increasing until a point p_2, then it is decreasing, so p_1 is the unique local minimum. $\diamond$

A.2 The d-dimensional Gaussian model

As before, we take $L \ge 0$, and we define

$$\tau_L = \inf\{t \mid R_t \notin [0, L]^d\}, \quad \tau = \inf\{t \mid R_t \notin (\mathbb{R}^+)^d\}$$

We notice that $\tau_L \to \tau$ a.s. as L tends to $+\infty$. First of all we search for a function h solution of $\mathcal{A}h = 0$, where $\mathcal{A}$ is the second order operator associated to $(R_t)_t$, that is:

$$\mathcal{A}h(x) = \langle A(c-x), Dh(x)\rangle + \frac{1}{2}\mathrm{Tr}\,(\sigma\sigma^* D^2 h(x)) \tag{A.2}$$

where Dh is the gradient of h and D^2h indicates the Hessian matrix of h. If h is such a function, then $M_t = h(R_{t\wedge\tau_L})$ is a bounded martingale (because $R_{t\wedge\tau_L} \in [0, L]^d$ and h is continuous), so it converges a.s.; but, if $\tau_L = +\infty$, then $h(R_{t\wedge\tau_L}) = h(R_t)$ cannot converge, so $\tau_L < +\infty$ a.s.; besides, since $(M_t)_t$ is a bounded martingale, we have $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ $\forall t \geq 0$, so:

$$h(x) = \mathbb{E}[h(R_0)] = \lim_{t\to+\infty} \mathbb{E}[h(R_{t\wedge\tau_L})] = \mathbb{E}\left[\lim_{t\to+\infty} h(R_{t\wedge\tau_L})\right] = \mathbb{E}[h(R_{\tau_L})]$$

where the third equality follows from Lebesgue theorem. We call $v(x, L) = \mathbb{P}\{R_{\tau_L}|_i = 0 \text{ for some } i\}$ under the condition $R_0 = x$. Then we have the intuitive conditions $v(0, L) = 1$ and $v(L, L) = 0$. Moreover we have that

$$\{\tau < +\infty\} = \{\exists L \text{ such that } R_{\tau_L}|_i = 0 \text{ for some } i\} = \bigcup_{L>0} \{R_{\tau_L}|_i = 0 \text{ for some } i\}$$

so

$$\mathbb{P}\{\tau < +\infty\} = \lim_{L\to+\infty} \{R_{\tau_L}|_i = 0 \text{ for some } i\} = \lim_{L\to+\infty} v(x, L) = 1$$

We have seen that $\mathbb{P}\{\tau < +\infty\} = 1$, that is there exists almost surely a T such that $r_T < 0$. Now, let's fix $T > 0$, and calculate $\mathbb{P}\{\tau \leq T\}$, that is the probability that the rates are positive up to the time T. First of all we give the result in terms of the solution of a parabolic PDE with boundary condition.

Lemma A.7 *If there exist a solution $v(t, x) \in C_b^1$ of the parabolic partial differential equation*

$$\begin{cases} \dfrac{\partial v}{\partial t} + \mathcal{A}v = 0 & (t, x) \in [0, T) \times (0, +\infty)^d \\ \\ v(T, x) = 0 & x \in (0, +\infty)^d \\ v(t, x) = 1 & t \in [0, T), x \in \partial(0, +\infty)^d \end{cases}$$

where $\mathcal{A}$ is given by Eq. (A.2), then $\mathbb{P}\{\tau \leq T\} = v(0, R_0)$.

Proof. We apply Itô formula to the function v:

$$dv(t, R_t) = \left(\frac{\partial v}{\partial t} + \mathcal{A}v\right)(t, R_t)\; dt + \frac{\partial v}{\partial x}(t, R_t)\sigma\; dW_t$$

then, because of the PDE, we have:

$$v(T \wedge \tau, R_{T\wedge\tau}) = v(0, r_0) + \int_0^{T\wedge\tau} \frac{\partial v}{\partial x}(t, R_t)\sigma\; dW_t$$

making the expectation, since $v \in C_b^1$, then the expectation of the stochastic integral is 0, and we get:

$$\begin{aligned} v(0,R_0) &= \mathbb{E}[v(T\wedge\tau, R_{T\wedge\tau})] = \mathbb{E}[v(T,R_T)\mathbf{1}_{\tau>T} + v(\tau,R_\tau)\mathbf{1}_{\tau\leq T}] = \\ &= \mathbb{E}[0\cdot\mathbf{1}_{\tau>T} + 1\cdot\mathbf{1}_{\tau\leq T}] = \mathbb{P}\{\tau\leq T\} \end{aligned}$$

$\diamond$

As before, the problem with this approach is that there is not an explicit solution to this PDE, so we have no idea of the behaviour of the solution.

A.3 Open problems

- generalize the two kinds of estimation of $\mathbb{P}\{\tau \leq T\}$ made for the Vasicek model to the d-dimensional Gaussian model
- make a similar work for the general Gaussian Musiela model, which is infinite dimensional