Chapter 4

Robustness of the Musiela model

4.1 A functional differential equation for interest rates derivatives

We start from the Musiela equation

$$\begin{cases} dr_t(x) = (Ar_t(x) + c(t, r_t(\cdot))(x) \; dt + \tau(t, r_t(\cdot))(x) \; dW_t, & t \in [0, T] \\ r_0(\cdot) \in H \end{cases} \tag{4.1}$$

where $A = \frac{\partial}{\partial x}$, $(W_t)_t$ is a d-dimensional Brownian motion, H is a separable Hilbert space of functions from $\mathbb{R}^+$ to $\mathbb{R}$ (for example, $L^2(\mathbb{R}^+)$ or $H^1(\mathbb{R}^+)$), $\tau : \mathbb{R}^+ \times H \to H^d$ and $c : \mathbb{R}^+ \times H \to H$, defined by $c(t, r(\cdot))(x) = \sum_{i=1}^d \tau_i(t, r(\cdot))(x) \int_0^x \tau_i(t, r(\cdot))(u) \; du$, satisfy the following conditions:

(A1) the mapping $c : \mathbf{R}_+ \times H \to H$ is measurable and

$$\|c(t, r) - c(t, r')\| \leq K\|r - r'\| \quad \forall r, r' \in H, \forall t \in [0, T]$$

(A2) the mapping $\tau : \mathbf{R}_+ \times H \to H^d$ is measurable and

$$\|\tau(t, r) - \tau(t, r')\| \leq K\|r - r'\| \quad \forall r, r' \in H, \forall t \in [0, T]$$

These conditions comprehend the stationary Gaussian case we analyzed in Chapter 1 as the particular case in which τ is a constant function.

Proposition 4.1 *If (A1) and (A2) hold then for every H-valued random variable r_0 independent of the Wiener process W there exists a unique continuous solution $(r_t)_t$ to (1.1) which is a strong Markov process on H. Moreover, if $\mathbb{E}[\|r_0\|^p] < \infty$ for a certain $p \geq 2$ then for any $T > 0$:*

$$\sup_{t \leq T} \mathbb{E}[\|r_t\|^p] \leq C_{T,p} \left(1 + \mathbb{E}[\|r_0\|^p]\right).$$

Moreover, if τ and c are Lipschitz transformations on $H^1(0, \infty)$ and $r_0 \in H^1(0, \infty)$ then $r_t \in H^1(0, \infty)$ $\mathbb{Q}$-almost surely for all $t \geq 0$.

The proof is an immediate consequence of the general theory developed in [5].

From now on we assume that our system is homogeneous in time, that is c and τ don't depend explicitly on time, but only on the rate $r \in H$. We know that the bonds' prices are deterministic functions of the spot forward rates:

$$B(t,T) = \mathcal{B}(t, r_t) = \exp\left(-\int_t^T r_t(u)\, du\right)$$

More in general, we can consider a zero coupon bond as the simplest case of an European contingent claim. We say that $F : H \to \mathbb{R}$ defines an **European interest rate derivative** or **European interest rate contingent claim** with maturity T if $\mathbb{E}[F(r_T)] < +\infty$; the claim gives to his owner the final payoff $F(r_T)$. In particular the case $F \equiv 1$ corresponds to the zero-coupon bond $B(t,T)$ with maturity T. Due to the Markov property of the process r the price C_T of the claim F at time $t \leq T$ is

$$\begin{aligned} C_T(t) &= \mathbb{E}\left[\exp\left(-\int_t^T r_u(0)\, du\right) F(r(T)) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^T r_u(0)\, du\right) F(r(T)) \middle| r(t)\right]. \end{aligned} \tag{4.2}$$

As in the finite dimensional case, we have the following result, that links the European contingent claim F with the equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,r) + Lu(t,r) - r(0)u(t,r) = 0 \\ \\ u(T,r) = F(r). \end{cases} \tag{4.3}$$

where L is defined on $D(A)$ by

$$LF(r) = \frac{1}{2}\langle D^2 F(r)\tau(r), \tau(r)\rangle + \langle Ar + c(r), DF(r)\rangle \tag{4.4}$$

Equation (4.3) is a Feynman-Kac equation in the Hilbert space H. To make our statements more precise, for $m \geq 0$ we define the space

$$C_m(H) = \left\{F : H \to \mathbf{R} \mid \frac{F(\cdot)}{1 + \|\cdot\|^m} \text{ bounded and uniformly continuous on } H\right\}$$

The space $C_m(H)$ is endowed with the norm

$$\|F\|_m = \sup_{r \in H}\left(\frac{|F(r)|}{1 + \|r\|^m}\right).$$

Let $C_m^n(H)$ denote the subspace of $C_m(H)$ containing all functions F which are n times Fréchet differentiable on H and

$$\sup_{r \in H}\left(\frac{\|D^k F(r)\|}{1 + \|r\|^m}\right) < \infty, \quad k = 1, \ldots, n,$$

where $D^k F(\phi)$ denotes the k-th Fréchet derivative of F. If $F \in C^1_m(H)$ then the derivative $DF(\phi)$ of F at $\phi \in H$ can be identified with an element of H and therefore $DF : H \to H$ is a continuous mapping. If $F \in C^2_m(H)$ then the second derivative $D^2F(\phi)$ is a symmetric linear operator in H and we can define a continuous mapping $D^2F : H \to \mathcal{L}(H)$ from H into the algebra of bounded operators on H. Finally by $C^n(H)$ we denote the vector space of all real-valued continuous functions on H with n continuous Fréchet derivatives. Let $\psi_1, \ldots, \psi_n \in \mathrm{dom}\,(A^*)$ and let P_n denote the orthogonal projection on the linear span H_n of the vectors $\psi_1, \ldots, \psi_n$. Moreover, let

$$\mathcal{D}_0 = \{F \in C_\alpha(H) : F = f \circ P_n,\, f \in C^2_m(R^n), n = 1, \ldots\}.$$

Note that every $F \in C_m(H)$ is a pointwise limit of elements of $\mathcal{D}_0$.

Theorem 4.2 *If τ and c are twice differentiable on H and such that*

$$\|\tau(\phi)\| + \|c(\phi)\| \leq C(1 + \|\phi\|).$$

then for every $F \in C^2_m(H)$ the function C_T is a unique solution of the Feynman-Kac equation (4.3) in the following sense.

i) The function $C_T : \mathbf{R}_+ \times H \to \mathbf{R}$ is bounded and continuous with respect to each variable.

ii) For every $t \geq 0$ we have $C_T(t, \cdot) \in C^2(H)$.

iii) $C_T \in C^1([0,T], H^1)$.

iv) Equation (4.3) holds for every $\phi \in$ dom (A) and $t \geq 0$. Moreover, C_T is given by (4.2).

For a proof, see [9].

This theorem is a first result, but it is not enough: in fact the principal examples of European interest rates derivatives (calls, puts, swaptions) are of class C^0 but not of class C^1.

4.2 Robustness of the Musiela model

We suppose that the dynamics of the forward rates is given by Equation

$$\begin{cases} dr_t(x) = (Ar_t(x) + \bar{c}_t(x))\; dt + \bar{\tau}_t(x)\; dW_t \\ r_0(\cdot) \in H \end{cases} \tag{4.5}$$

where $A = \frac{\partial}{\partial x}$, $(W_t)_t$ is a d-dimensional Brownian motion, H is a separable Hilbert space of functions from $\mathbb{R}^+$ to $\mathbb{R}$ (for example, $L^2(\mathbb{R}^+)$ or $H^1(\mathbb{R}^+)$), $\bar{\tau}_t(x)$ is a H^d-valued adapted process, and $\bar{c}(x) = \sum_{i=1}^d \bar{\tau}^i_t(x) \int_0^x \bar{\tau}^i_t(u)\; du$ is a H-valued process; moreover, the process τ is supposed to take values in some compact set $\Sigma \in H^d$. This model is a quite general one, and it is an analogous of stochastic volatility models for stock prices analyzed in literature in the last years (see for example [8] and [1]).

Now we consider an operator in the market that wants to create and sell an European interest rate derivative asset with payoff $F(r_{\bar{T}})$ depending on the value in the maturity $\bar{T}$ of all the forward rates curve $r_{\bar{T}}$. In many relevant cases, the payoff F will depend not on all the curve $r_{\bar{T}}$, but only on the values of N particular bonds with maturity $T_i > \bar{T}$, so it will be of the form

$$F(B(\bar{T},T_1),\ldots,B(\bar{T},T_N)), \quad \bar{T} < T_1 < \ldots < T_N \tag{4.6}$$

From now on, we suppose that F is of the form (4.6). In order to hedge this asset, the operator chooses to use the homogeneous Gaussian Musiela model described in Chapter 1. Then he (she) builds a self-financing strategy in this way: he (she) fixes the price C_t of the option at time t as

$$C_t = u(t, r_t)$$

where u is the solution of the following nonlinear functional differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,r) + \sup\limits_{\tau\in\Sigma} Lu(t,r) - r(0)u(t,r) = 0, \quad (t,r) \in [0,\bar{T}] \times H \\ \\ u(\bar{T},r) = F(r), \quad r \in H \end{cases} \tag{4.7}$$

where L is the operator (4.4). Equation (4.7) is a Hamilton-Jacobi-Bellmann type equation; we notice that its finite-dimensional versions appear for the first time in one dimension in [1], and it is generalized to the finite-dimensional case in [14]; as in the previous works Equation (4.7) will be called from now on the **Black-Scholes-Barenblatt** equation associated to the contingent claim F.

Lemma 4.3 *If F is of the form (4.6), and the function $u(t,r)$ is a solution of Equation (4.7) and is of the form*

$$u(t,r) = C(t, r_t(0), B(t,T_1), \ldots, B(t,T_N))$$

where $C : [0,\bar{T}] \times \mathbb{R} \times \mathbb{R}^N$, $r_t(0)$ is the spot rate at time t and $B(t,T_i)$ are the values in t of the zero-coupon bonds maturing in T_i, $i = 1,\ldots,N$, then C is solution of the following equation

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,r_0,B) + r_0\langle DC(t,r_0,B),B\rangle + \dfrac{1}{2}\sup\limits_{\tau\in\Sigma} \mathrm{tr}\ \left(D^2C(t,r_0,B)(\bar{B}\Gamma)(\bar{B}\Gamma)^*\right) = r_0 C(t,B), \\ \\ C(\bar{T},r_0,B) = F(B), \quad r_0 \in \mathbb{R}, B \in \mathbb{R}^N \end{cases} \tag{4.8}$$

where $DC(t,r_0,B) = (\frac{\partial C}{\partial B_1}(t,r_0,B),\ldots,\frac{\partial C}{\partial B_n}(t,r_0,B))$, and $D^2C(t,r_0,B) = (\frac{\partial^2 C}{\partial B_i B_j}(t,r_0,B))_{ij}$, $\Gamma(t,T)$ is a $d \times N$ matrix which the i-th row is the volatility $\Gamma(t,T_i)$ of the bond $B(t,T_i)$, given by Equation (3.1), and

$$\bar{B} = \mathrm{diag}\ (B) = \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & B_n \end{pmatrix}$$

Proof. We notice that

$$\frac{\partial u}{\partial r} = \frac{\partial C}{\partial B}\frac{\partial B}{\partial r}, \qquad \frac{\partial^2 u}{\partial r^2} = \left(\frac{\partial B}{\partial r}\right)^* \frac{\partial^2 C}{\partial B^2}\left(\frac{\partial B}{\partial r}\right) + \frac{\partial C}{\partial B}\frac{\partial^2 B}{\partial r^2}$$

and that for all maturities T_i we have:

$$\frac{\partial B(t,T_i)}{\partial r} = -\mathbf{1}_{[0,T_i-t]}B, \qquad \frac{\partial^2 B(t,T_i)}{\partial r^2} = \mathbf{1}^*_{[0,T_i-t]}B\mathbf{1}_{[0,T_i-t]}$$

then from Equation (4.7) we get:

$$\begin{aligned}
r(0)u(t,r) &= \frac{\partial u}{\partial t}(t,r) + \max_{\tau\in\Sigma} Lu(t,r) = \\
&= \frac{\partial u}{\partial t}(t,r) + \max_{\tau\in\Sigma}\left(\left\langle \frac{\partial u}{\partial r}(t,r), Ar + c(r)\right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 u}{\partial r^2}(t,r)\tau(r),\tau(r)\right\rangle\right) = \\
&= \frac{\partial C}{\partial t}(t,B) + \max_{\tau\in\Sigma}\left(-\sum_{i=1}^{N}\frac{\partial C}{\partial B_i}(t,B)B(t,T_i)\langle\mathbf{1}_{[0,T_i-t]}, Ar + c(r)\rangle + \right.\\
&\quad +\frac{1}{2}\sum_{i=1}^{N}\tau(r)\left(\mathbf{1}^*_{[0,T_i-t]}B(t,T_i)\frac{\partial^2 C}{\partial B^2}(t,B)B(t,T_i)\mathbf{1}_{[0,T_i-t]} + \right.\\
&\quad \left.\left. +\frac{\partial C}{\partial B}\mathbf{1}^*_{[0,T_i-t]}B_i\mathbf{1}_{[0,T_i-t]}\right)\tau(r)\right) = \\
&= \frac{\partial C}{\partial t}(t,B) + \max_{\tau\in\Sigma}\left(-\sum_{i=1}^{N}\frac{\partial C}{\partial B_i}(t,B)B(t,T_i)(r(T_i-t) - r(0) + \langle\mathbf{1}_{[0,T_i-t]}, c(r)\rangle) + \right.\\
&\quad +\frac{1}{2}\mathrm{Tr}\left(\sum_{i=1}^{N}\Gamma(t,T_i)^* B(t,T_i)\frac{\partial^2 C}{\partial B^2}(t,B)B(t,T_i)\Gamma(t,T_i) + \right.\\
&+ \quad \left.\left.\Gamma(t,T_i)^*\frac{\partial C}{\partial B_i}B(t,T_i)\Gamma(t,T_i)\right)\right) = \\
&= \frac{\partial C}{\partial t}(t,B) + \max_{\tau\in\Sigma}\left(\sum_{i=1}^{N}\frac{\partial C}{\partial B_i}(t,B)B(t,T_i)r(0) + \right.\\
&\quad \left. + \frac{1}{2}\mathrm{Tr}\left(\sum i = 1^N \Gamma(t,T_i)^* B(t,T_i)\frac{\partial^2 C}{\partial B^2}(t,B)B(t,T_i)\Gamma(t,T_i)\right)\right)\\
&\quad \left(-\sum_{i=1}^{N}\frac{\partial C}{\partial B_i}(t,B)B(t,T_i)(r(T_i-t))\right)
\end{aligned}$$

and Equation (4.8) follows. ◇

Equation (4.8) will be called from now on the **reduced Black-Scholes-Barenblatt equation** for F.

In order to hedge the claim F, the agent builds the hedging portfolio holding the quantity Δ_t^i of the zero-coupon bond $B(t,T_i)$ in the hedging portfolio Π as

$$\Delta_t^i = \frac{\partial C}{\partial B_i}(t, B(t,T))$$

where we indicated as $B(t,T)$ the vector $(B(t,T_1),\ldots,B(t,T_N))$; Π follows the dynamics:

$$\begin{cases} d\Pi_t = r_t(0)(\Pi_t - \langle \Delta_t, B(t,T)\rangle)\ dt + \langle \Delta_t, dB(t,T)\rangle \\ \Pi_0 = P_0 \end{cases}$$

(for a proof, see [11]).

We define the **tracking error**:

$$e_t = \Pi_t - C_t$$

with $e_0 = 0$. Intuitively, the tracking error gives the error made by the operator in his (her) estimation, or better the difference between the hedging portfolio held by the agent and the option he (she) sold.

We want to find a superstrategy where e_t is increasing $\forall t$ $\mathbf{Q}$-a.s.

Theorem 4.4 *If C is a $C^{1,2}([0,T)\times\mathbf{R}^n)$ solution of Equation (4.8), then $(e_t)_t$ is an increasing process.*

Proof. The evolution of $(e_t)_t$ is the following:

$$\begin{aligned}
de_t &= r_t(0)(\Pi_t - \langle \Delta_t, B(t,T)\rangle)\ dt + \langle \Delta_t, dB(t,T)\rangle - \frac{\partial C}{\partial t}(t,B(t,T))\ dt - \\
&\quad -\langle DC(t,B(t,T)), dB(t,T)\rangle - \\
&\quad -\frac{1}{2}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\Gamma(t,T))(\bar{B}(t,T)\Gamma(t,T))^*\right)\ dt = \\
&= r_t(0)(\Pi_t - \langle \Delta_t, B(t,T)\rangle)\ dt + r_t(0)\langle B(t,T), DC(t,B(t,T))\rangle\ dt - r_t(0)C_t\ dt + \\
&\quad +\frac{1}{2}\max_{\tau\in\Sigma}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\Gamma(t,T))(\bar{B}(t,T)\Gamma(t,T))^*\right)\ dt - \\
&\quad -\frac{1}{2}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\Gamma(t,T))(\bar{B}(t,T)\Gamma(t,T))^*\right)\ dt = \\
&= r_t(0)e_t\ dt + \frac{1}{2}\max_{\tau\in\Sigma}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\Gamma(t,T))(\bar{B}(t,T)\Gamma(t,T))^*\right) - \\
&\quad -\frac{1}{2}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\sigma(t,T))(\bar{B}(t,T)\sigma(t,T))^*\right) = \\
&= r_t(0)e_t\ dt + R_t\ dt
\end{aligned}$$

where

$$\begin{aligned}
R_t &= \frac{1}{2}\max_{\tau\in\Sigma}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\Gamma(t,T))(\bar{B}(t,T)\Gamma(t,T))^*\right) - \\
&\quad -\frac{1}{2}\mathrm{tr}\ \left(D^2C(t,B(t,T))(\bar{B}(t,T)\sigma(t,T))(\bar{B}(t,T)\sigma(t,T))^*\right) \geq 0
\end{aligned}$$

We call the quantity R_t **consumption rate**. We have:

$$e_t = M_t \int_0^t \frac{1}{M_u} R_u\ du$$

where $M_t = \exp(\int_0^t r_u(0)\ du)$. Then, to have an increasing e_t is equivalent to have $R_t \geq 0$. Since this is verified, we have the thesis. $\diamond$

4.3 Sensibility of the model

First of all we suppose that the rates evolve following the Vasicek model:

$$dr_t = a(b - r_t)\ dt + \sigma\ dW_t$$

The zero coupon bond is

$$B(t,T) = e^{-(T-t)Y(t,T)} = \mathbb{E}\left[e^{-\int_t^T r_u\ du}|\mathcal{F}_t\right]$$

The explicit solution for the spot rate is:

$$r_u = e^{a(u-t)} + \int_0^{u-t} e^{a(u-t-v)} ab\ dv + \int_0^{u-t} e^{a(u-t-v)}\sigma\ dW_v$$

that is, r_u depends linearly on r_t, on a deterministic term and on a (Gaussian) random variable independent on $\mathcal{F}_t$, so we can represent the zero coupon bond in this way:

$$B(t,T) = \exp\left(-r_t(T-t)\beta(T-t) - (T-t)d(T-t)\right)$$

where

$$\beta(T-t) = \frac{1}{T-t}\int_t^T e^{a(u-t)}\ du = \frac{e^{a(T-t)}-1}{a(T-t)}$$

Then we have:

$$\begin{aligned} Y(t,T) &= r_t\beta(T-t) + d(T-t) \\ dY(t,T) &= \beta(T-t)\ dr_t - \beta'(T-t)r_t\ dt - d'(T-t)\ dt \end{aligned}$$

By applying Itô formula, we obtain that the volatility of $B(t,T)$ is

$$\Gamma(t,T) = (T-t)\beta(T-t)\sigma$$

Now we take a portfolio having the deterministic payoffs F_i in the dates T_i, $i = 1,\dots,n$; then its evolution is given by:

$$\begin{aligned} P_t &= \sum_{i=1}^n F_i B(t,T_i) \\ dP_t &= \sum_{i=1}^n F_i dB(t,T_i) = r_t P_t\ dt + \sum_{i=1}^n F_i B(t,T_i)\Gamma(t,T_i)\ dW_t \end{aligned}$$

Then the volatility of P_t is

$$\frac{\sum_{i=1}^n F_i B(t,T_i)\Gamma(t,T_i)}{\sum_{i=1}^n F_i B(t,T_i)} = \frac{\sum_{i=1}^n F_i B(t,T_i)(T_i-t)\beta(T_i-t)\sigma}{P_t}$$

Now we define the **duration** of the portfolio P with respect to the direction f in this way:

$$D_f P = \frac{\sum_{i=1}^n F_i B(t,T_i)(T_i-t)f(T_i-t)}{P_t}$$

then the volatility of P_t can be interpreted as the duration of the portfolio with respect to the direction $f(x) = \sigma\beta(x)$.

There is a link with the classical definition of duration: in fact we have that

$$\frac{1}{P_t}\frac{\partial P_t}{\partial r_t} = \frac{1}{P_t}\sum_{i=1}^{n} F_i \frac{\partial B(t,T_i)}{\partial r_t} = \frac{\sum_{i=1}^{n} F_i B(t,T_i)(T_i - t)\beta(T_i - t)\sigma}{P_t}$$

If more in general we have that $Y(t,T)$ is a deterministic function of k reference quantities (which can be rates or other variables)

$$Y(t,T) = \beta_1(t,T)R_t^1 + \ldots + \beta_k(t,T)R_t^k + \alpha(t,T)$$

then:

$$\frac{1}{P_t}\frac{\partial P_t}{\partial R_t^j} = \frac{1}{P_t}\sum_{i=1}^{n} F_i \frac{\partial B(t,T_i)}{\partial R_t^j} = \frac{\sum_{i=1}^{n} F_i B(t,T_i)(T_i - t)\beta_j(t,T_i)}{P_t}$$

that can be interpreted as the **sensibility** of P_t in the j-th direction.

4.4 Open problems

- existence and uniqueness of the solution of the BSB equation (4.7); while in the case in which the final condition $F \in C^2(H)$ the proof would probably be quite standard, in the case $F \in C^0$ is not clear; probably one has to introduce a notion like viscosity solutions in infinite dimensions.

- is it possible to reduce the sup operator in both the equations (4.7) and (4.8) to a max? Moreover, when is it possible to project the $\Sigma \in H^\infty$ in a compact set in $\mathbb{R}^N$? Finally, is it possible to reduce the reducted BSB equation (4.8) to a partial differential equation in $\mathbb{R}^{N+2}$ (the $N+2$ variables are N bonds, the short rate and the time)?

- if $Y(t,T) = G(r_t(\cdot))$ is a linear functional of all the forward rate curve $r_t(\cdot)$, is it possible to write something like

$$\frac{1}{P_t}\frac{\partial P_t}{\partial f} = \frac{\sum_{i=1}^{n} F_i B(t,T_i)(T_i - t)\frac{\partial G}{\partial f}}{P_t}$$

  where $\frac{\partial G}{\partial f}$ is the Frechet derivative of G with respect to F?