Introduction

The aim of this work is to analyze the Musiela model for the forward rates presented in [13], and in particular the Gaussian version of this model. The Musiela model is a reparametriziation of the HJM model [10], so it is essentially a set of relations which an interest rate model must satisfy in order not to have arbitrage opportunities in the market.

This work is intended to be an attempt to analyze this model, and in particular what happens if we try to approximate the forward rate dynamics with a finite or infinite dimensional Gaussian model. Various topics associated to this problem are touched, as the reduction of the model to a finite dimensional one, the calibration of the model to hystorical market datas in order to get the most possibly precise shape of the forward rate curve, and the so called "robustness" of the model, that is a kind of good dependence of the model from the extimated datas, in such a way that even if an agent extimates the coefficients of the model making errors, he (she) succeeds in obtaining a superhedging strategy, and that a little error on the extimation of the curve gives a little error on the price and the replication of the derivative assets. In this work we wanted to present ideas related to this topics, giving also open problems, without insisting too much on technicalities; this also because, while some parts (like the reduction of the model to a finite dimensional one) don't present technical difficulties, other ones (in particular the part on the robustness of the model) would require heavy mathematical tools, whose complexity would be parhaps sufficient for a Ph. D. thesis! Finally there is an appendix devoted to the aspect that is considered on of the main drawbacks of the Gaussian models, that is the possibility of negative rates; also in this appendix we try to present the main ideas and don't insist too much on the technical side.

In the work we proceed in this way. First we present the Musiela model: its core is the stochastic differential equation

$$\begin{cases} dr(t,x) = \left(\dfrac{\partial}{\partial x} r(t,x) + \displaystyle\sum_{n=1}^{\infty} \tau_n(t,x) \int_0^x \tau_n(t,u)\ du \right)\ dt + \displaystyle\sum_{n=1}^{\infty} \tau_n(t,x)\ dw_t^n, \quad t \geq 0 \\ \\ r(0,\cdot) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+) \text{ given} \end{cases}$$

where $r(t,\cdot) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+)$ $\mathbb{Q}$-a.s. $\forall t$, and $(w^n)_n$ are independent real Brownian motions; we often will write the equation in a shorter way:

$$\begin{cases} dr_t = (Ar_t + c_t)\ dt + \{\tau_t, dW_t\}, \quad t \geq 0 \\ r_0 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+) \text{ given} \end{cases}$$

We notice that the choice of τ determines entirely the evolution of the forward rate; in particular we can choose to consider a finite or infinite number of driving Brownian motions, or we can choose to consider a deterministic τ, thus obtaining a Gaussian model. We also notice that the general model is in the space $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$, which is a locally convex space; in this kind of spaces the theory of stochastic integration is rather difficult, while it becomes much simpler in Hilbert spaces. Though such a problem can be treated in its generality, here we present a simpler problem, that is to analyze the model supposing that the forward rates take values in some Hilbert spaces contained in $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$. We present some results from an existing paper [15], regarding the existence and uniqueness of a so called mild solution of the Musiela equation in the simplest case, that is when τ is deterministic and there is a finite number of driving Brownian motions; the problems is solved in two different Hilbert spaces, namely $H^1_\gamma(\mathbb{R}^+)$ and $H^1_\gamma(\mathbb{R}^+)$, in which the equation has different behaviour. We present also some results linked to the existence and (possibly) the uniqueness of an invariant measure for the equation. This last problem is linked to the calibration of the model: we will find out that an implicit calibration gives very few informations on the model, while a particular hystorical calibration succeeds in giving much more information. But each time that a calibration based on time series is used (as in our case), there is always the implicit hypothesis that the process is stationary; this is equivalent to the existence of an invariant measure for the driving equation.

This study introduces us to the main topic of this work: even if we restrict the model to the Gaussian case, the natural framework for it is $W^{1,1}_{\text{loc}}(\mathbb{R}^+)$, and the choices of the two Hilbert spaces appears arbitrary; moreover, there is no apparent reason to impose that there is only a finite number of driving Brownian motions. The main question is then: is it possible to find a “natural” Hilbert space in which the equation “behaves well”? More formally, does it exist a Hilbert space and an orthonormal basis $(e_n)_n$ such that, if we project the solution of the equation on the first N components, we make a small error. The simplest case in which this happens is when the model is itself finite dimensional, so we first analyze how the Musiela equation becomes when we have the Vasicek model (the simplest Gaussian model) or more generally a finite dimensional Gaussian model in which the state variables are particular reference rates. Starting from this, we make some qualitative considerations on which could be the “right” Hilbert space for the equation, but these considerations are neither rigorous nor fully satisfactory! We finish this session with a first attempt to calculate the error we make when we approximate the “real” dynamics of the model with a finite dimensional approximation. This leaves a first series of open problems, linked to the choices of the space, of the basis, and of the criterium to use to calculate the error.

Now we come to the part about the calibration of the model: we present an implicit approach, that is classically used in the market, but doesn't give us much information; then we present two hystorical approaches, and we develop with more detail the first one; in particular, we apply the method known as “Karhunen–Loeve expansion”. This method is classically used in fluidodynamics in order to find an orthonormal basis, whose vectors are known as “empirical orthonormal functions”, which diagonalize the covariance operator of an infinite dimensional random variable; it turns out that this method gives nice numerical results, in the sense that few components of the basis are sufficient to describe much of the evolution of the system, and there doesn't exist another orthonormal basis that can improve the result. We present the method following [12], and we give a way to apply the method to our situation, considering a time series as a sample of an infinite dimensional random variable having as law an invariant measure of the equation. Here there is a hidden detail: in fact we always studied the Musiela model under the risk-neutral probability $\mathbb{Q}$, while we can obtain from the time series an estimation of the hystorical probability $\mathbb{P}$. This doesn't create problems, because since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, the Gaussian random variable we are studying has the same covariance both under $\mathbb{P}$ as under $\mathbb{Q}$, so we can proceed safely.

Finally we arrive to the part devoted to the robustness of the model. First of all we present a functional differential equation that gives, under no arbitrage assumptions, the price of an interest rate derivative under the hypothesis that the forward rates follow a (even non linear) Musiela model. We do this for two reason: the first is to make a link between this case and the classical finite dimensional cases (see [4], [16]) in which the price of interest rate derivatives can be derived via a PDE; besides, we can introduce the mathematical tools that we need later and an equation that is simpler, but similar, to the one known as Black–Scholes–Barenblatt equation, that we'll meet later. Then we consider the real robustness problem: we suppose that the dynamics of the forward rates is given by the Musiela equation, but that we aren't able to determine precisely the process τ, but we only know that τ takes values in some compact set $\Sigma \subseteq H^d$. This model is a quite general one, and it is an analogous of the stochastic volatility models for stock prices analyzed in literature in the last years (see for example [8] and [1]). Now we consider an operator in the market that wants to create and sell an European interest rate derivative asset with payoff $F(r_{\bar{T}})$ depending on the value in the maturity $\bar{T}$ of all the forward rates curve $r_{\bar{T}}$. In many relevant cases, the payoff F will depend not on all the curve $r_{\bar{T}}$, but only on the values of N particular bonds with maturity $T_i > \bar{T}$, so it will be of the form

$$F(B(\bar{T},T_1),\dots,B(\bar{T},T_N)), \quad \bar{T} < T_1 < \dots < T_N$$

From now on, we suppose that F is of the form above. In order to hedge this asset, the operator chooses to use the homogeneous Gaussian Musiela model described in Chapter 1. Then he (she) builds a self-financing strategy in this way: he (she) fixes the price C_t of the option at time t as

$$C_t = u(t, r_t)$$

where u is the solution of a nonlinear functional differential equation, namely a Hamilton-Jacobi-Bellmann type equation;

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,r) + \sup\limits_{\tau\in\Sigma} Lu(t,r) - r(0)u(t,r) = 0, \quad (t,r) \in [0,\bar{T}] \times H \\ \\ u(\bar{T},r) = F(r), \quad r \in H \end{cases}$$

where L is a particular second order operator acting on functionals on the Hilbert space H. We remark that the finite-dimensional versions of this equation appears for the first time in one dimension in [1], and it is generalized to the finite-dimensional case in [14]; as in the previous works the Equation above will be called from now on the **Black-Scholes-Barenblatt** equation associated to the contingent claim F. To our knowledge, this is the first time that such an approach is used in hedging interest rates derivatives, so it is also the first time that such an equation is analyzed in this context. In order to hedge the claim F, the agent builds the hedging portfolio holding the quantity Δ_t^i of the zero-coupon bond $B(t,T_i)$ in the hedging portfolio Π defined in the classical delta-hedging way as the derivative of the solution of the equation with respect to the corresponding asset. Finally we define the tracking error $e_t = \Pi_t - C_t$, with $e_0 = 0$. Intuitively, the tracking error gives the error made by the operator in his (her) estimation, or better the difference between the hedging portfolio held by the agent and the option he (she) sold. We find out that if C is a $C^{1,2}([0,T) \times \mathbf{R}^n)$ solution of a suitable finite dimensional version of the BSB equation, then $(e_t)_t$ is an increasing process, that is the delta-hedging gives a superstrategy for our operator. This is the part where the open problems are more deep, not only because of the intrinsic difficulty of the study of the BSB equation in a Hilbert space, but also because all the cases of interest in Finance are such that the final payoff is not even C^1; it seems so that it would be necessary to develop some tools analogous to what viscosity solution are in finite dimensions.

The appendix is devoted to one of the main drawbacks of the Gaussian models, that is the possibility of negative rates with positive probability. First of all we concentrate on the one-dimensional Vasicek model, and calculate the probability that at some time there will be a negative forward rate; this probability being 1, we calculate the probability that there will be a negative forward rate before a fixed instant T. While we succeed to calculate exactly the first probability in terms of the solution of a ODE, we can't calculate explicitly the second probability, because it is given in terms of the solution to a PDE, that doesn't appear to have an explicit solution. Then we give two extimations of this probability, one using Kolmogorov's maximal inequality and the fact that the process $(e^{at}r_t)_t$ is a martingale, and the other one making a time change, seeing the stochastic integral appearing in the solution r_t as a Brownian motion with respect to the new time scale, and using a sort of Cameron-Martin theorem. Then, we try to do a similar work for the d-dimensional Gaussian model, even if the work in this direction is far to be complete. Anyway, the final aim of this appendix would be to extend all or some of the results found in the case of Vasicek's model to the general Gaussian Musiela model in a Hilbert space: from a mathematical point of view, we feel that this kind of problem is interesting and far from being trivial.

The work is organized as follows. In Chapter 1 there is an introduction to the Musiela model, with the main technical results concerning the equation in the spaces $H^1_\gamma(\mathbb{R}^+)$ and $H^1(\mathbb{R}^+)$. In Chapter 2 we want to evaluate when the homogeneous Gaussian Musiela model can be reduced to a finite factor model; moreover, we want to find a Hilbert space H which behaves well with respect to this interpretation. When this is not possible, we want to evaluate the error we do when we approximate a term structure having a Musiela dynamics with a finite factor model. In Chapter 3 we concentrate on the derivation of the forward curve from economical datas; in particular, we present an "implicit" approach, that derives bonds' volatility from the prices of derivatives in the market, and two "historical" ones, that derive bonds' volatility from the prices of bonds in the past. In particular, we develop one hystorical approach using the Karhunen–Loeve expansion. In Chapter 4 we present the functional differential equation satisfied by the bonds' prices under no arbitrage assumptions. Then we present the Black–Scholes–Barenblatt equation, and a finite dimensional version of this equation, and prove that in certain cases the strategy built starting from the BSB equation is a superhedging one. Finally we present the concept of extended duration and show the possible links of this concept with our model. At last in the Appendix 1 we concentrate on the positivity of interest rates; first we calculate the probability to have a negative rate for the Vasicek model, then we make the same calculations in the case of a Gaussian d-dimensional model.

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