Robustness for path-dependent volatility models

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Abstract

In this paper we consider a generalisation of the Hobson-Rogers model proposed by Foschi and Pascucci in [10] for financial markets where the evolution of the prices of the assets depends not only on the current value but also on past values. Using differentiability of stochastic processes with respect to the initial condition, we analyse the robustness of such a model with respect to the so-called offset function, which generally depends on the entire past of the risky asset and is thus not fully observable. In doing this, we extend previous results of [3] to contingent claims which are globally Lipschitz with respect to the price of the underlying asset, and we improve the dependence of the necessary observation window on the maturity of the contingent claim, which now becomes of linear type, while in [3] it was quadratic. Finally, in this framework we give a characterisation of the stationarity assumption used in [3], and prove that this model is stationary if and only if it is reduced to the original Hobson-Rogers model. We conclude by calibrating the model to the prices of two indexes using two different volatility shapes.

Keywords: path-dependent volatility models, Hobson-Rogers model, differential of stochastic processes, Lagrange theorem.
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1 Introduction

The Black and Scholes model is based upon the assumption that the behaviour of the logarithm of the asset price is well represented by a Gaussian process with

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stationary independent increments. This assumption is formulated mathematically by imposing that the drift and the volatility are deterministic functions. The important role played by the volatility in the Black-Scholes formula and the fact that a constant volatility assumption is not consistent with observations of actual financial markets are both well known, and for these reasons several proposals have been made to introduce some sort of stochastic dependency in the volatility parameter, either with a deterministic dependency on the current stock price or with a dedicated dynamics driven by a new source of uncertainty.

One of the models which better fits market data is the so-called Hobson-Rogers model, introduced in [12] and studied with respect to various features in [1, 3, 5, 7, 8, 9, 10, 17, 20]. The Hobson-Rogers model consists in the following (we only present the version with a single offset function). For a risky asset whose price is denoted by the process $S = (S_t)_{t\in\mathbb{R}}$, define the discounted log-price process Z_t at time t as $Z_t = \log(S_t e^{-rt})$ where r is the (constant) risk-free interest rate, and the *offset function* of order 1, denoted by $P = (P_t)_t$, by

$$P_t = \int_0^\infty \lambda e^{-\lambda u}(Z_t - Z_{t-u})du \tag{1}$$

the constant λ being a parameter of the model which describes the rate at which past information is discounted. One then assumes that Z satisfies the SDE (stochastic differential equation)

$$dZ_t = -\frac{1}{2}\sigma^2(P_t)dt + \sigma(P_t)\ dW(t) \tag{2}$$

where $\sigma(\cdot)$ is a strictly positive function and $(W_t)_{t\in\mathbb{R}}$ is a so-called two-sided Brownian motion [4] under a risk-neutral probability measure $\mathbb{P}$ (see [2, 3, 12] and the references therein for details). Thus, the dynamics of S turns out to be

$$dS_t = rS_t\ dt + \sigma(P_t)S_t\ dW_t$$

This model is seen as a "good" model because no new Brownian motions (or other sources of uncertainty) have been introduced in the specification of the price process. This means that the market is complete and any contingent claim is hedgeable in this way: if we calculate the stochastic differential of P, we obtain

$$dP_t = dZ_t - \lambda P_t\ dt \tag{3}$$

so (Z, P), as well as (S, P), is a 2-dimensional Markov process. Thus one can employ the Kolmogorov equation when pricing a contingent claim with final payoff $h(S_T)$. In fact, its price $V_t = \mathbb{E}[e^{-r(T-t)}h(S_T)|\mathcal{F}_t]$ is of the form $V_t = F(t, S_t, P_t)$, where F is the solution of the Kolmogorov equation

$$\begin{cases} F_t + rsF_s - \lambda pF_p + \left(\frac{1}{2}s^2F_{ss} + sF_{ps} + \frac{1}{2}F_{pp} - \frac{1}{2}F_p\right)\sigma^2(p) = rF \\ F(p, s, T) = h(s) \end{cases} \tag{4}$$

An alternative approach is to calculate the price of contingent claims with the Monte Carlo method, see [1] for details.

The main theoretical drawback of this model is that, in order to find the present offset P_0 and thus implement the technique above, one has to know in principle the whole past trajectory of S, which is impossible in practice, or to infer P_0 in some other way. This can be done in three different ways.

1. One can regard P_0 as an unknown parameter, which can be calibrated from market prices of liquid derivatives together with λ and the other parameters of σ.

2. If $p \to \sigma(p)$ is invertible, then it is possible to infer P_0 from the quadratic variation of S, as $\frac{d}{dt}\langle S\rangle_{0+} = S_0\sigma^2(P_0)$.

3. As in [3], one can use the model with a misspecification $\widetilde{\Sigma}_0 := (\widetilde{P}_0, Z_0)$ instead of the true initial values $\Sigma_0 := (P_0, Z_0)$ (notice that $Z_0 = \log S_0$ is observed in the market and thus not misspecified) and use robustness estimates to control the error of this misspecification.

While the first two methods betray the spirit of this model, *i.e.* the dependence of the volatility on past values and in particular on the offset, which is also empirically tested in [17] with positive results, they also raise their own problems. The first method introduces another parameter to calibrate, and numerical tests performed by the authors found that with this additional "parameter" the calibration becomes very unstable and sometimes does not deliver any result at all. With regard to the second method, even if σ is easily invertible, one has to calculate the quadratic variation from discrete data and perform a numerical derivative, both operations being numerically unstable: this easily delivers a misspecified $\widetilde{P}_0$ as a result, thus making it necessary to recur to the robustness estimates of the third method. Moreover, in some cases (see for example Subsection 6.1 and the original paper [12]), at some points σ is not even invertible.

For the reasons above, we chose to concentrate on the third method, which we explain in more detail here. By using a misspecified $\widetilde{\Sigma}_0 := (\widetilde{P}_0, Z_0)$ instead of the true (but unobservable) $\Sigma_0 := (P_0, Z_0)$, we obtain, as solution of Equations (2) and (3), a misspecified process $\widetilde{\Sigma}_t := (\widetilde{P}_t, \widetilde{Z}_t)$ instead of the "true" process $\Sigma_t := (P_t, Z_t)$; we then search for the initial condition $\widetilde{P}_0$ which minimizes the error of pricing the contingent claim $h(S_T)$. This approach has been carried out in detail in [3] via L^2-estimates of the solutions of Equations (2) and (3) with respect to the initial condition, and the result in that paper is that

$$\mathbb{E}\left[\sup_{0\le u\le T} |\Sigma_u - \widetilde{\Sigma}_u|^2\right] \le K\mathbb{E}[|P_0 - \widetilde{P}_0|^2]\mathrm{e}^{cT^2+dT}$$

where K, c and d are suitable constant depending on λ and the function σ. The L^2-error of P_0 is then estimated by linking it to the L^2-error of P_{-R}, where $R > 0$ is assumed to be an observation interval of the past price of the stock $(S_t)_{t\in[-R,0]}$, which we assume to be available, and one has $\mathbb{E}[|P_0 - \widetilde{P}_0|^2] = \mathrm{e}^{-\lambda R}\mathbb{E}[|P_{-R} - \widetilde{P}_{-R}|^2]$. This latter L^2-error is then assumed to be equal to the variance V of the invariant measure of P: in fact, if the dynamics (3) of P is ergodic, then we have that $\mathbb{E}[|P_{-R} - \widetilde{P}_{-R}|^2]$ converges to V as $R \to +\infty$, so if R is big enough we can use V to approximate $\mathbb{E}[|P_{-R} - \widetilde{P}_{-R}|^2]$.

This entails that when pricing a European (possibly path-dependent) contingent claim with maturity T and final payoff $h(S(\cdot))$ we have

$$\left|\mathbb{E}[h(S_T)] - \mathbb{E}[h(\widetilde{S}_T)]\right|^2 \le KJ^2\mathrm{e}^{-\lambda R}V\mathrm{e}^{cT^2+dT} \tag{5}$$

where J is the Lipschitz constant of the functional $z(\cdot) \to h(\mathrm{e}^{z(\cdot)})$, provided h enjoys the latter Lipschitz condition (i.e., it is Lipschitz with respect to the log-return), which is quite non-standard. For example, a call option does not satisfy this condition and so in that case one must resort to call-put parity in order to obtain a robustness result. Furthermore, if one wants to obtain prices with a given precision, the estimate (5) gives a quadratic dependence of R on the maturity T, which produces very long and unlikely observation times: in the examples in [3], for a maturity of $T = 3$ months one has to observe $R \simeq 4$ years of the historical prices of S, while for a maturity of $T = 5$ years this observation window becomes $R \simeq 100$ years long. If h is a simple European claim, then an analogous estimate holds which has all the previous drawbacks.

The aim of this paper is threefold.

- First, we extend the previous analysis to a slight generalisation of the Hobson-Rogers model, carried out by Foschi and Pascucci in [10], which introduces a more flexible dependence of the offset function on time.

- Second, we succeed in presenting an L^1-estimates (instead of the L^2-estimates of [3]) of the form

$$\mathbb{E}\left[\sup_{0\le u\le T} |\Sigma_u - \widetilde{\Sigma}_u|\right] \le K(T)\mathbb{E}[|P_0 - \widetilde{P}_0|]\mathrm{e}^{dT} \tag{6}$$

  where K is a function with subexponential growth. This is done by obtaining the differential $\partial\Sigma_t$ of the sample paths of Σ with respects to the initial condition Σ_0 and using Lagrange's theorem, which entails $\Sigma_t^{(p,z)} - \Sigma_t^{(\widetilde{p},z)} = \int_{\widetilde{p}}^{p} \partial_1\Sigma_t^{(\zeta,z)}\,\mathrm{d}\zeta$. Since for $\partial_1\Sigma_t^{(\zeta,z)}$ we obtain estimates of the kind $\mathbb{E}[\|\partial_1\Sigma_t^{(\zeta,z)}\|] \le K\mathrm{e}^{dt}$, by integrating we get the desired estimate (6), which is a notable improvement of the result in [3].

- Third, we extend the above results to functions h which are Lipschitz in the natural variable S and do not require artificial Lipschitz conditions in auxiliary variables.

The paper is organised as follows. In Section 2 we present the generalisation of the Hobson-Rogers model by Foschi and Pascucci [10]. In Sections 3 we present an L^1-estimate based on the differentiation of the stochastic process Σ with respect to the initial condition P_0, while in Section 4 we present an L^1-estimate on the pricing error on a contingent claim which is Lipschitz continuous with respect to the natural variable S. In Section 5 we extend the results of [3] on the use of past information and the width of the observation window R to the present case. In Section 6 we characterise the conditions under which P is ergodic, and the main result is that this is possible only if the Foschi-Pascucci model reduces to the Hobson-Rogers model; two practical examples of $\sigma(\cdot)$ are then presented and our estimates for R are compared with the corresponding estimates in [3]. In the final Section 7 we calibrate the model to the prices of two indexes (Eurostoxx50 and IBEX) using two different volatility shapes for each index.

2 The Foschi-Pascucci model

Let $T > 0$ and $\varphi : (-\infty, T] \to [0, +\infty)$ be a piecewise continous function, integrable on $(-\infty, T]$ and such that $\varphi > 0$ on $[0, T]$, and define

$$\Phi(t) := \int_{-\infty}^{t} \varphi(s)\,\mathrm{d}s \ .$$

We denote the stock price by S and use Z to denote the log-discounted price process $Z_t := \ln(\mathrm{e}^{-rt} S_t)$. Let us consider the process

$$M_t := \frac{1}{\Phi(t)} \int_{-\infty}^{t} \varphi(s) Z_s\,\mathrm{d}s, \quad t \in]0, T]$$

which has stochastic differential

$$\mathrm{d}M_t = \frac{\varphi(t)}{\Phi(t)} (Z_t - M_t)\,\mathrm{d}t = \frac{\varphi(t)}{\Phi(t)} D_t\,\mathrm{d}t$$

where $D_t := Z_t - M_t$, whose stochastic differential is thus given by

$$\mathrm{d}D_t = \mathrm{d}Z_t - \frac{\varphi(t)}{\Phi(t)} D_t\,\mathrm{d}t \ . \tag{7}$$

For Z we assume the dynamics

$$\mathrm{d}Z_t = -\frac{1}{2}\sigma^2(D_t)\,\mathrm{d}t + \sigma(D_t)\,\mathrm{d}W_t \tag{8}$$

under an[1] equivalent martingale measure $\mathbb{P}$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is Hölder and such that there exists a unique strong solution of the system (7–8). It is clear that, under these assumptions the processes (Z, M) and (Z, D) (as well as the processes (S, M) and (S, D)) are Markovian. Besides, the model is arbitrage-free and complete, and in particular the price at time 0 of a European contingent claim with payoff $h(S_T, M_T)$ at time T is given by $F(0, S_0, M_0)$, where F is the solution of the Kolmogorov equation

$$\begin{cases} F_t + rsF_s - \dfrac{\varphi(t)}{\Phi(t)} dF_d + \left(\dfrac{1}{2}s^2 F_{ss} + sF_{ds} + \dfrac{1}{2}F_{dd} - \dfrac{1}{2}F_d\right)\sigma^2(d) = rF \\[2ex] F(s, d, T) = h(s, \log s - rT - d) \end{cases} \tag{9}$$

A wise choice of φ allows one to obtain, with this more general approach, cases already studied in literature:

- the choice $\varphi(t) = \mathrm{e}^{\lambda t}$ reduces this model to the original Hobson-Rogers model;
- the choice $\varphi(t) = \mathbf{1}_{[0,T]}(t)$ allows one to obtain prices for Asian options written on the geometric mean;

[1] or better, "the" equivalent martingale measure, as it turns out that this market is complete.

- the choice $\frac{\varphi}{\Phi}(t)$ piecewise linear on $[0,T]$, which corresponds to $\varphi(t) = (at+b)\mathrm{e}^{\frac{a}{2}t^2+bt+c}$ piecewise on $[0,T]$, gives enough flexibility to calibrate well to market data (see [10] and below).

Empirical results in [10] show that this model, using the last specification for φ, is comparable to both the Hobson-Rogers model and to the Heston model with regard to the replication error of discrete-time hedging portfolios; in particular, this model always lies between the other two but is nearer to the better one, in both quiet market and nervous scenarios.

3 First L^1-estimate

Our aim is now to obtain L^1-estimates on $\Sigma := (D, Z)$ by the use of differentiation of stochastic processes and Lagrange's theorem. The use of this latter technique and the requirement for an L^1-estimate will allow us to obtain log-linear estimates like (6) instead of the original log-quadratic ones present in [3].

The starting point is to see that the process $\Sigma := (D, Z)$ is differentiable with respect to the initial value, and that the derivative process with respect to D_0 satisfies the SDE in the following theorem.

Theorem 1. *Assume that*

- *σ and σ^2 are differentiable, with locally Lipschitz derivatives bounded respectively by L_1 and L_2;*
- *φ/Φ is differentiable, locally Lipschitz, and such that $0 < \underline{\lambda} \le \frac{\varphi}{\Phi}(t) \le \bar{\lambda}$ for $t \in [0,T]$, with $\underline{\lambda} := \inf_t \left|\frac{\varphi}{\Phi}(t)\right|$ and $\bar{\lambda} := \sup_t \left|\frac{\varphi}{\Phi}(t)\right|$,*

and call $\Sigma = \Sigma^{D_0,Z_0}$ the solution to Equations (7–8) with initial condition $\Sigma_0 := (D_0, Z_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^2)$. If $(D_0, Z_0) = (d, z) \in \mathbb{R}^2$, then $\Sigma^{d,z}$ is differentiable with respect to the initial value, and the derivative process with respect to $D_0 = d$ satisfies the SDE

$$\begin{cases} d\partial_1 D_t = -\left(\frac{1}{2}\left(\sigma^2\right)'(D_t) + \frac{\varphi(t)}{\Phi(t)}\right)\partial_1 D_t\, dt + \sigma'(D_t)\,\partial_1 D_t\, dW_t \\ d\partial_1 Z_t = -\frac{1}{2}\left(\sigma^2\right)'(D_t)\,\partial_1 D_t\, dt + \sigma'(D_t)\,\partial_1 D_t\, dW_t \end{cases} \tag{10}$$

with initial conditions $\partial_1 D_0 = 1$, $\partial_1 Z_0 = 0$, where for a generic process $X = D, Z$ we set $\partial_1 X_t := \partial X_t^{d,z}/\partial d$.

Proof. Rewrite Equations (7–8) in the form

$$\begin{cases} \mathrm{d}Z_t = -\frac{1}{2}\sigma^2(D_t)\,\mathrm{d}t + \sigma(D_t)\,\mathrm{d}W_t \\ \mathrm{d}D_t = \mathrm{d}Z_t - \frac{\varphi(\tau_t)}{\Phi(\tau_t)}D_t\,\mathrm{d}t \\ \mathrm{d}\tau_t = \mathrm{d}t \end{cases} \tag{11}$$

with initial conditions $Z_0 = z$, $D_0 = d$, $\tau_0 = 0$: it is then clear that (D, Z, τ) is a homogeneous Markov process. By applying [18, Theorem V.39, p. 305], we obtain the result. □

Remark 1. *We can easily solve the SDE (10). In fact, the SDE relative to $\partial_1 D$ is linear, so that*

$$\partial_1 D_t = \exp\left[\int_0^t \left(-\frac{\varphi(s)}{\Phi(s)} - \frac{1}{2}\left(\sigma^2\right)'(D_s) - \frac{1}{2}\left(\sigma'\right)^2(D_s)\right) ds + \int_0^t \sigma'(D_s)\, dW_s\right]$$
$$= \frac{\Phi(0)}{\Phi(t)} \exp\left(\int_0^t -\frac{1}{2}\left(\sigma^2\right)'(D_s)\, ds\right) Y_t$$

where we denote

$$Y_t \quad = \quad \exp\left(\int_0^t \sigma'(D_s)\, dW_s - \frac{1}{2}\int_0^t \left(\sigma'\right)^2(D_s)\, ds\right) ,$$

Notice that, since σ' is bounded, Y is a positive martingale, bounded in L^p on $[0,T]$. It is then sufficient to integrate to obtain $\partial_1 Z_t$.

We now present a L^1-estimate of the kind of Equation (6). As we are dealing with the 2-dimensional processess Σ and $\partial_1\Sigma$, we will use the norm $\mathbb{E}[\|\cdot\|_1]$, where $\|x\|_1 := |x_1| + |x_2|$ for all $x \in \mathbb{R}^2$.

Theorem 2. *In the framework given above, the following inequalities hold:*

$$\mathbb{E}\left[\sup_{0\le u\le t} \|\partial_1 \Sigma_u\|_1\right] \quad\le\quad \left(5 + \bar{\lambda} t\right) \mathrm{e}^{\left(\frac{L_1^2}{2} + \frac{L_2}{2} - \underline{\lambda}\right)^+ t}$$
$$\mathbb{E}\left[\|\partial_1 \Sigma_t\|_1\right] \quad\le\quad \left(3 + \bar{\lambda} t\right) \mathrm{e}^{\left(\frac{L_2}{2} - \underline{\lambda}\right)^+ t} .$$

Proof. By Remark 1, it follows that

$$\mathbb{E}\left[\partial_1 D_t\right] = \frac{\Phi(0)}{\Phi(t)} \mathbb{E}\left[\mathrm{e}^{-\int_0^t \frac{1}{2}\left(\sigma^2\right)'(D_s)\,\mathrm{d}s} Y_t\right] = \mathrm{e}^{-\underline{\lambda} t + \frac{1}{2} L_2 t} \mathbb{E}\left[Y_t\right] \le \mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)t} ,$$

where

$$\frac{\Phi(0)}{\Phi(t)} = \exp(\log\Phi(0) - \log\Phi(t)) = \exp\left(-\int_0^t \frac{\varphi(s)}{\Phi(s)}\,\mathrm{d}s\right) \le \mathrm{e}^{-\underline{\lambda} t} .$$

Now, write Equation (7) in the form

$$Z_t = Z_0 + D_t - D_0 + \int_0^t \frac{\varphi(s)}{\Phi(s)} D_s\,\mathrm{d}s .$$

By differentiating under the integral sign (see Proposition 1 in Appendix A), we get

$$\partial_1 Z_t = \partial_1 D_t - 1 + \int_0^t \frac{\varphi(s)}{\Phi(s)} \partial_1 D_s\,\mathrm{d}s .$$

Then we have

$$\mathbb{E}\left[\sup_{0\le u\le t} \|\partial_1 \Sigma_u\|_1\right] \le \mathbb{E}\left[\sup_{0\le u\le t} \partial_1 D_u\right] + \mathbb{E}\left[\sup_{0\le u\le t} |\partial_1 Z_u|\right] \le$$
$$\le \quad 1 + 2\mathbb{E}\left[\sup_{0\le u\le t} \partial_1 D_u\right] + \int_0^t \frac{\varphi(s)}{\Phi(s)} \mathbb{E}\left[\partial_1 D_s\right]\,\mathrm{d}s$$
$$\le \quad 1 + 2\mathbb{E}\left[\sup_{0\le u\le t} \frac{\Phi(0)}{\Phi(u)} \mathrm{e}^{-\int_0^u \frac{1}{2}\left(\sigma^2\right)'(D_s)\,\mathrm{d}s} Y_s\right] + \bar{\lambda}\int_0^t \mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)s}\,\mathrm{d}s$$
$$\le \quad 1 + 2\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)t} \mathbb{E}\left[\sup_{0\le u\le t} Y_s\right] + \bar{\lambda}\int_0^t \mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)s}\,\mathrm{d}s$$

By applying Doob's inequality we have

$$\mathbb{E}\left[\sup_{0\le u\le t} Y_s\right] \le \mathbb{E}\left[\sup_{0\le u\le t} Y_s^2\right]^{1/2} \le \left(4\mathbb{E}[Y_t^2]\right)^{1/2}$$

while

$$\mathbb{E}\left[Y_u^2\right] = \mathbb{E}\left[\mathrm{e}^{-\int_0^u (\sigma')^2(D_s)\,\mathrm{d}s + 2\int_0^u \sigma'(D_s)\,\mathrm{d}W_s}\right] = \mathbb{E}\left[\mathrm{e}^{\int_0^u (\sigma')^2(D_s)\,\mathrm{d}s}\widetilde{Y}_u\right] \le \mathrm{e}^{L_1^2 u}\mathbb{E}\left[\widetilde{Y}_u\right]$$

where

$$\widetilde{Y}_u = \exp\left(-2\int_0^u (\sigma')^2(D_s)\,\mathrm{d}s + 2\int_0^u \sigma'(D_s)\,\mathrm{d}W_s\right) .$$

Since $\widetilde{Y}$ is a positive martingale, we have $\mathbb{E}\left[Y_u^2\right] \le \mathrm{e}^{L_1^2 u}$. Finally we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le u\le t} \|\partial_1 \Sigma_u\|_1\right] &\le 1 + 4\mathrm{e}^{\left(\frac{1}{2}L_1^2 + \frac{1}{2}L_2 - \underline{\lambda}\right)t} + \bar{\lambda}\int_0^t \mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)s}\,\mathrm{d}s \\
&\le 1 + 4\mathrm{e}^{\left(\frac{1}{2}L_1^2 + \frac{1}{2}L_2 - \underline{\lambda}\right)^+ t} + \bar{\lambda}t\mathrm{e}^{\left(\frac{1}{2}L_1^2 + \frac{1}{2}L_2 - \underline{\lambda}\right)^+ t} .
\end{aligned}$$

Analogously,

$$\begin{aligned}
\mathbb{E}\left[\|\partial_1 \Sigma_t\|_1\right] &= \mathbb{E}\left[\partial_1 D_t\right] + \mathbb{E}\left[|\partial_1 Z_t|\right] \le 1 + 2\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)t} + \bar{\lambda}\int_0^t \mathbb{E}\left[\partial_1 D_s\right]\,\mathrm{d}s \\
&\le 1 + 2\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)t} + \bar{\lambda}\int_0^t \mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)s}\,\mathrm{d}s \le \\
&\le 1 + 2\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)t} + \bar{\lambda}t\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)^+ t} \le \\
&\le \left(3 + \bar{\lambda}t\right)\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)^+ t} .
\end{aligned}$$

□

The next theorem now follows easily.

Theorem 3. *Under the assumptions of Theorem 1, for each initial conditions* η, $\widetilde{\eta} \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ *and* $z \in \mathbb{R}$, *the following inequalities hold*

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le u\le t}\left\|\Sigma_u^{(\eta,z)} - \Sigma_u^{(\widetilde{\eta},z)}\right\|_1\right] &\le \left(5 + \bar{\lambda}t\right)\mathrm{e}^{\left(\frac{1}{2}L_1^2 + \frac{1}{2}L_2 - \underline{\lambda}\right)^+ t}\mathbb{E}\left[|\eta - \widetilde{\eta}|\right] , \\
\mathbb{E}\left[\left\|\Sigma_t^{(\eta,z)} - \Sigma_t^{(\widetilde{\eta},z)}\right\|_1\right] &\le \left(3 + \bar{\lambda}t\right)\mathrm{e}^{\left(\frac{1}{2}L_2 - \underline{\lambda}\right)^+ t}\mathbb{E}\left[|\eta - \widetilde{\eta}|\right] .
\end{aligned}$$

Proof. At first, let $\eta = d$ and $\widetilde{\eta} = \widetilde{d}$ be constant. For the first inequality, by the Fundamental Theorem of Calculus and the results of Theorem 2 we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le r\le t}\left\|\Sigma_r^{(d,z)} - \Sigma_r^{(\widetilde{d},z)}\right\|_1\right] = \mathbb{E}\left[\sup_{0\le r\le t}\left\|\int_{\widetilde{d}}^{d} \partial_1 \Sigma_r^{(\zeta,z)}\,\mathrm{d}\zeta\right\|_1\right] \le \\
&\le \int_{d\wedge\widetilde{d}}^{d\vee\widetilde{d}} \mathbb{E}\left[\sup_{0\le r\le t}\left\|\partial_1 \Sigma_r^{(\zeta,z)}\right\|_1\right]\mathrm{d}\zeta \le \left(5 + \bar{\lambda}t\right)\mathrm{e}^{\left(\frac{1}{2}L_1^2 + \frac{1}{2}L_2 - \underline{\lambda}\right)^+ t}|d - \widetilde{d}| ,
\end{aligned}$$

which is the first inequality. For the second inequality proceed analogously, once again using the results of Theorem 2.

For the case when η and $\widetilde{\eta}$ are not constants but random variables in $L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, it is sufficient to consider the conditional expectation with respect to $\mathcal{F}_0$ and use the tower property:

$$\mathbb{E}\left[\sup_{0\le r\le t}\left\|\Sigma_r^{(\eta,z)}-\Sigma_r^{(\widetilde{\eta},z)}\right\|_1\right]=\mathbb{E}\left[\mathbb{E}\left[\sup_{0\le r\le t}\left\|\Sigma_r^{(\eta,z)}-\Sigma_r^{(\widetilde{\eta},z)}\right\|_1|\mathcal{F}_0\right]\right]\le$$
$$\le \mathbb{E}\left[\left(5+\bar{\lambda}t\right)\mathrm{e}^{(\frac{1}{2}L_1^2+\frac{1}{2}L_2-\underline{\lambda})^+t}|\eta-\widetilde{\eta}|\right]=\left(5+\bar{\lambda}t\right)\mathrm{e}^{(\frac{1}{2}L_1^2+\frac{1}{2}L_2-\underline{\lambda})^+t}\mathbb{E}[|\eta-\widetilde{\eta}|]$$

and proceed analogously for the second inequality. □

We conclude this section by applying the results of Theorem 3 to the pricing error of European derivative assets $h(S(\cdot))$, possibly path-dependent, which are Lipschitz with respect to the log-return, i.e. such that the map $C^0([0,T]) \ni f \to h(\mathrm{e}^{f(\cdot)}) \in \mathbb{R}$ is globally Lipschitz. In the case of a European claim which is a function of the final price, we require that $\mathbb{R} \ni x \to h(\mathrm{e}^x) \in \mathbb{R}$ is globally Lipschitz. Some examples of such assets follow (see [3] for details and proofs).

- European put: the payoff is $h(s(\cdot)) = (K - s_T)^+$. Then the Lipschitz constant of $x \to h(\mathrm{e}^x)$ in this case is K.
- European call: the payoff is now given by $h(s(\cdot)) = (s_T - K)^+$. We can use the put-call parity and write this expression as $h(S(\cdot)) = S_T - K - (K - S_T)^+$, so the error is the same as when pricing the put.
- Asian put: the payoff is now given by $h(s(\cdot)) = (K - \int_0^T s(t)\,dt)^+$. It can be proved that the Lipschitz constant of $C^0([0,T]) \ni f \to h(\mathrm{e}^{f(\cdot)}) \in \mathbb{R}$ is in this case $2K$.
- Lookback put: the payoff is now given by $h(s(\cdot)) = \left(K - \max_{0\le t\le T} s(t)\right)^+$.

  In this case it can be proved that the Lipschitz constant from $C^0([0,T])$ to $\mathbb{R}$ is K.

As announced in the Introduction, these new estimates are of the kind $|\mathbb{E}[h(S(\cdot))] - \mathbb{E}[h(\widetilde{S}(\cdot))]| \le J(T)\mathrm{e}^{dT}\mathbb{E}[|D_0 - \widetilde{D}_0|]$, where J is a function with sub-exponential growth: we thus improve the results of [3], where the dependence on T was of the type $|\mathbb{E}[h(S(\cdot))] - \mathbb{E}[h(\widetilde{S}(\cdot))]| \le J(T)\mathrm{e}^{cT^2+dT}\mathbb{E}[|D_0 - \widetilde{D}_0|]$.

Theorem 4. *Suppose that the coefficients $d \mapsto \sigma(d)$ and $d \mapsto \sigma^2(d)$ admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively.*

1. *Let $h : C^0[0,T] \to \mathbb{R}$ be the payoff of a claim such that the functional $C^0[0,T] \to \mathbb{R} : f \mapsto h(\mathrm{e}^f)$ is globally Lipschitz (with respect to the sup-norm $\|\cdot\|_{C^0}$), with Lipschitz constant J. Then*

$$\left|\mathbb{E}\left[h\left(S(\cdot)\right)\right]-\mathbb{E}[h(\widetilde{S}(\cdot))]\right|\le J\left(5+\bar{\lambda}T\right)\mathrm{e}^{\left(\frac{L_1^2}{2}+\frac{L_2}{2}-\underline{\lambda}\right)^+T}\mathbb{E}\left[\left|D(0)-\widetilde{D}(0)\right|\right] \tag{12}$$

2. *Let $h : \mathbb{R} \to \mathbb{R}$ be the payoff of a European claim such that the function $\mathbb{R} \to \mathbb{R} : x \mapsto h(\mathrm{e}^x)$ is globally Lipschitz with constant J. Then*

$$\left|\mathbb{E}\left[h\left(S(T)\right)\right]-\mathbb{E}[h(\widetilde{S}(T))]\right|\le J\left(3+\bar{\lambda}T\right)\mathrm{e}^{\left(\frac{L_2}{2}-\underline{\lambda}\right)^+T}\mathbb{E}\left[\left|D(0)-\widetilde{D}(0)\right|\right] \tag{13}$$

Proof. By (12) we have

$$\begin{aligned}\left|\mathbb{E}\left[h\left(S(\cdot)\right)\right]-\mathbb{E}[h(\widetilde{S}(\cdot))]\right| &\leq \mathbb{E}\left[\left|h\left(S(\cdot)\right)-h\left(\widetilde{S}(\cdot)\right)\right|\right]=\mathbb{E}\left[\left|h\left(\mathrm{e}^{Z(\cdot)}\right)-h\left(\mathrm{e}^{\widetilde{Z}(\cdot)}\right)\right|\right]\\ &\leq J\mathbb{E}\left[\left\|Z(\cdot)-\widetilde{Z}(\cdot)\right\|_{C^0}\right]=J\mathbb{E}\left[\sup_{0\leq t\leq T}\left|Z(t)-\widetilde{Z}(t)\right|\right]\\ &\leq J\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\Sigma(t)-\widetilde{\Sigma}(t)\right\|_1\right],\end{aligned}$$

and we conclude by recalling the first inequality of Theorem 3. For (13), the proof is similar and we conclude by using the second inequality of Theorem 3. □

4 Estimates for globally Lipschitz contingent claims

The results of the previous sections only allow one to obtain pricing errors of derivative assets which are Lipschitz with respect to the log-return, condition which is rather unusual in the financial literature. As a simple example, notice that a plain vanilla call option $h(S(T)) := (S(T)-K)^+$ does not satisfy the previous Lipschitz condition, although it is globally Lipschitz. As seen in the previous section, this example can be easily taken care of by using put-call parity for vanilla options. There are, however, examples where this is not possible although the function h is globally Lipschitz with respect to the natural variable S: consider, for example, of a floating strike Asian option, with payoff $h(S(\cdot)) := (S_T - \frac{1}{T}\int_0^T S_t\, dt)^+$.

In order to extend our analysis to contingent claims which are globally Lipschitz in the natural variable S, we now present other L^1-estimates for the variables (D,S) and for their differentials with respect to the (still misspecified) initial condition D_0. This will produce error estimates for globally Lipschitz contingent claims.

Lemma 1. *Under the assumptions of Theorem 1, and if in addition $|\sigma|$ is bounded by a constant M, then*

$$\mathbb{E}\left[S_t^2\right] \leq s^2 \mathrm{e}^{(2r+M^2)t}$$

where $S_0 \equiv s$, and

$$\mathbb{E}\left[\sup_{0\leq u\leq t} S_u^2\right] \leq 4s^2\mathrm{e}^{(2r+M^2)t}$$

Proof. By applying the Itô formula, we have

$$\mathrm{d}S_t^2 = 2S_t^2\sigma(D_t)\,\mathrm{d}W_t + S_t^2(2r+\sigma^2(D_t))\,\mathrm{d}t$$

Denoting by $\mathcal{M}^2$ the Hilbert space of all progressively measurable (with respect to the completed natural filtration of the Brownian motion) and square-integrable (with respect to each product space $\Omega\times[0,t]$, for every $t>0$) random variables, since $S_t^2\sigma(D_t)\in\mathcal{M}^2$, we have

$$\mathbb{E}[S_t^2] = \mathbb{E}[S_0^2] + \mathbb{E}\left[\int_0^t S_u^2(2r+\sigma^2(D_u))\,\mathrm{d}u\right] \leq s^2 + \int_0^t (2r+M^2)\mathbb{E}[S_u^2]\,\mathrm{d}u$$

By applying Gronwall's lemma we obtain the first inequality. For the second inequality, since S is a martingale it is sufficient to apply Doob's inequality and obtain

$$\mathbb{E}\left[\sup_{0\leq u\leq t} S_u^2\right] \leq 4\mathbb{E}[S_t^2] \leq 4s^2 \mathrm{e}^{(2r+M^2)t} \ .$$

□

Remark 2. *By Theorem 1 we know that the flow $(d,z) \mapsto Z_t^{(d,z)}$ is continously differentiable. Hence clearly so too is the the flow $(d,s) \mapsto S_t = \mathrm{e}^{Z^{(d,\ln s)}}$, and in particular*

$$\partial_1 S_t = \partial_1 Z_t \cdot S_t$$

where $s = \mathrm{e}^z$ is the condition at time 0 for S.

Lemma 2. *Under the assumptions of Theorem 1, we have $(\partial_1 D_t)^2 \in \mathcal{M}^2$, and*

$$\mathbb{E}\left[(\partial_1 D_t)^2\right] \leq \mathrm{e}^{\beta t}$$

where $\beta := \left(L_1^2 + L_2 - 2\underline{\lambda}\right)^+$.

Proof. Applying the Itô formula, one verifies that $\left(\partial_1 D_t\right)^2$ solves the SDE

$$\begin{cases} \mathrm{d}\left(\partial_1 D_t\right)^2 = \left(-\left(\sigma^2\right)'(D_t) - 2\dfrac{\varphi(t)}{\Phi(t)} + \left(\sigma'\right)^2(D_t)\right)\left(\partial_1 D_t\right)^2 \,\mathrm{d}t + 2\sigma'\left(\partial_1 D_t\right)^2 \mathrm{d}W_t \\ \left(\partial_1 D_0\right)^2 = 1 \end{cases} \tag{14}$$

Moreover, since the coefficients are bounded, it easily follows that $\left(\partial_1 D\right)^2 \in \mathcal{M}^2$. Let us prove the asserted inequality. By (14), we have

$$\begin{aligned} \mathbb{E}\left[(\partial_1 D)_t^2\right] &= 1 + \mathbb{E}\left[\int_0^t (\partial_1 D)_s^2 \left(-2\frac{\varphi(s)}{\Phi(s)} - \left(\sigma^2\right)'(D_s) + \left(\sigma'\right)^2(D_s)\right) \mathrm{d}s\right] \\ &\quad + \mathbb{E}\left[\int_0^t 2\left(\partial_1 D_s\right)^2 \sigma'(D_s)\,\mathrm{d}W_s\right] \\ &= 1 + \mathbb{E}\left[\int_0^t (\partial_1 D)_s^2 \left(-2\frac{\varphi(s)}{\Phi(s)} - \left(\sigma^2\right)'(D_s) + \left(\sigma'\right)^2(D_s)\right) \mathrm{d}s\right] \\ &\leq 1 + \left(L_1^2 + L_2 - 2\underline{\lambda}\right)^+ \int_0^t \mathbb{E}\left[(\partial_1 D)_s^2\right] \mathrm{d}s \ . \end{aligned}$$

Finally, we conclude by applying Gronwall's lemma. □

Theorem 5. *Under the assumptions of Lemma 1, we have*

$$\mathbb{E}\left[\sup_{0\leq u\leq t} |\partial_1 S_u|\right] \leq s\mathrm{e}^{\frac{2r+M^2+\beta}{2}t}\left(L_2 t + 4L_1\sqrt{t}\right)$$

and

$$\mathbb{E}\left[|\partial_1 S_t|\right] \leq s\mathrm{e}^{\frac{2r+M^2+\beta}{2}t}\left(\frac{L_2}{2}t + L_1\sqrt{t}\right)$$

where $\beta := \left(L_1^2 + L_2 - 2\underline{\lambda}\right)^+$.

Proof. Let us prove the first inequality. By Remark 2, and remembering the dynamics for $\partial_1 Z_t$ in Equation (10), we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le u\le t}|\partial_1 S_u|\right] &\le \mathbb{E}\left[\sup_{0\le u\le t} S_u\left\|\int_0^u \sigma'(D_s)\partial_1 D_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^u \left(\sigma^2\right)'(D_s)\partial_1 D_s\,\mathrm{d}s\right\|\right]\\
&\le \mathbb{E}\left[\sup_{0\le u\le t}\frac{S_u}{2}\left|\int_0^u \left(\sigma^2\right)'(D_s)\partial_1 D_s\,\mathrm{d}s\right|\right] + \mathbb{E}\left[\sup_{0\le u\le t} S_u\left|\int_0^u \sigma'(D_s)\partial_1 D_s\,\mathrm{d}W_s\right|\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\left(\sup_{0\le u\le t} S_u\right)\int_0^t\left|\left(\sigma^2\right)'(D_s)\partial_1 D_s\right|\,\mathrm{d}s\right] + \mathbb{E}\left[\left(\sup_{0\le u\le t} S_u\right)\left(\sup_{0\le u\le t}\left|\int_0^u \sigma'(D_s)\partial_1 D_s\,\mathrm{d}W_s\right|\right)\right]\\
&\le \frac{1}{2}\left(\mathbb{E}\left[\sup_{0\le u\le t} S_u^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\int_0^t\left|\left(\sigma^2\right)'(D_s)\partial_1 D_s\right|\,\mathrm{d}s\right)^2\right]\right)^{1/2}\\
&\quad + \left(\mathbb{E}\left[\sup_{0\le u\le t} S_u^2\right]\right)^{1/2}\left(\mathbb{E}\left[\sup_{0\le u\le t}\left|\int_0^u \sigma'(D_s)\partial_1 D_s\,\mathrm{d}W_s\right|^2\right]\right)^{1/2}\\
&\le \frac{1}{2}\cdot 2s\mathrm{e}^{\frac{2r+M^2}{2}t}\left(\mathbb{E}\left[t\int_0^t\left(\left(\sigma^2\right)'(D_s)\right)^2\left(\partial_1 D_s\right)^2\,\mathrm{d}s\right]\right)^{1/2}\\
&\quad +2s\mathrm{e}^{\frac{2r+M^2}{2}t}\cdot 2\left(\mathbb{E}\left[\int_0^t\left(\sigma'(D_s)\partial_1 D_s\right)^2\,\mathrm{d}s\right]\right)^{1/2}\\
&\le s\mathrm{e}^{\frac{2r+M^2}{2}t}\left(\sqrt{t}L_2 + 4L_1\right)\left(\int_0^t \mathbb{E}[(\partial_1 D_s)^2]\,\mathrm{d}s\right)^{1/2}\\
&\le s\mathrm{e}^{\frac{2r+M^2}{2}t}\left(\sqrt{t}L_2 + 4L_1\right)\sqrt{\int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}s}\\
&\le s\mathrm{e}^{\frac{2r+M^2+\beta}{2}t}\left(L_2 t + 4L_1\sqrt{t}\right)
\end{aligned}$$

The second inequality can be proved in an analogous way using the estimate for $\mathbb{E}[S_t^2]$ of Lemma 1 instead of Doob's inequality. □

Theorem 6. *Under the assumptions of Lemma 1, for each η, $\widetilde{\eta}\in L^2$ and $s\in\mathbb{R}$, the following inequalities hold:*

$$\mathbb{E}\left[\sup_{0\le u\le t}\left|S_u^{(\eta,s)} - S_u^{(\widetilde{\eta},s)}\right|\right] \le s\mathrm{e}^{\frac{2r+M^2+\beta}{2}t}\left(L_2 t + 4L_1\sqrt{t}\right)\mathbb{E}\left[|\eta-\widetilde{\eta}|\right]$$

and

$$\mathbb{E}\left[\left|S_t^{(\eta,s)} - S_t^{(\widetilde{\eta},s)}\right|\right] \le s\mathrm{e}^{\frac{2r+M^2+\beta}{2}t}\left(\frac{L_2}{2}t + L_1\sqrt{t}\right)\mathbb{E}\left[|\eta-\widetilde{\eta}|\right]$$

Proof. Rewrite the proof of Theorem 3, using the inequalities of Theorem 5 □

Now we present error estimates for globally Lipschitz contingent claims. The main results of these estimates are still of the kind $|\mathbb{E}[h(S(\cdot))] - \mathbb{E}[h(\widetilde{S}(\cdot))]| \le J(T)\mathrm{e}^{dT}\mathbb{E}[|D_0 - \widetilde{D}_0|]$, where J is a function with sub-exponential growth: this means that also in this situation we improve the results of [3].

Theorem 7. *Suppose $d\mapsto\sigma(d)$ and $d\mapsto\sigma^2(d)$ admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively (hence σ and σ^2 are sublinear), and suppose that $|\sigma|$ is bounded by M.*

1. *Let $h : C^0[0,T] \to \mathbb{R}$ be a path-dependent payoff function, globally Lipschitz with constant J. Then we have*

$$\left|\mathbb{E}\left[\mathrm{e}^{-rT}h(S(\cdot))\right] - \mathbb{E}\left[\mathrm{e}^{-rT}h(\widetilde{S}(\cdot))\right]\right| \le Js\mathrm{e}^{\frac{M^2+\beta}{2}T}\left(L_2T + 4L_1\sqrt{T}\right)\mathbb{E}\left[\left|D_0 - \widetilde{D}_0\right|\right]. \tag{15}$$

2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a European payoff function, globally Lipschitz with constant J. Then we have*

$$\left|\mathbb{E}\left[\mathrm{e}^{-rT}h(S_T)\right] - \mathbb{E}\left[\mathrm{e}^{-rT}h(\widetilde{S}_T)\right]\right| \le Js\mathrm{e}^{\frac{M^2+\beta}{2}T}\left(\frac{L_2}{2}T + L_1\sqrt{T}\right)\mathbb{E}\left[\left|D_0 - \widetilde{D}_0\right|\right] . \tag{16}$$

Proof. For the first inequality, notice that

$$\begin{aligned}\left|\mathbb{E}\left[\mathrm{e}^{-rT}h(S(\cdot))\right] - \mathbb{E}[\mathrm{e}^{-rT}h(\widetilde{S}(\cdot))]\right| &\le \mathbb{E}\left[\mathrm{e}^{-rT}\left|h(S(\cdot)) - h(\widetilde{S}(\cdot))\right|\right] \le J\mathrm{e}^{-rT}\mathbb{E}\left[\left\|S(\cdot) - \widetilde{S}(\cdot)\right\|_{C^0}\right]\\ &= J\mathrm{e}^{-rT}\mathbb{E}\left[\sup_{0\le t\le T}\left|S(t) - \widetilde{S}(t)\right|\right]\end{aligned}$$

and conclude applying the first inequality of Theorem 6. To prove the second inequality, observe that the inequalities

$$\left|\mathbb{E}\left[\mathrm{e}^{-rT}h(S_T)\right] - \mathbb{E}[\mathrm{e}^{-rT}h(\widetilde{S}_T)]\right| \le \mathbb{E}\left[\mathrm{e}^{-rT}\left|h(S_T) - h(\widetilde{S}_T)\right|\right] \le J\mathrm{e}^{-rT}\mathbb{E}\left[\left|S_T - \widetilde{S}_T\right|\right] ,$$

allow one to conclude applying the second inequality of Theorem 6. □

5 Using past information

The aim of the L^1-estimates of the previous sections is to choose $\widetilde{P}_0$ in order to minimise the final error. As in [3] we assume as known all the past values of the price S_t for $t \in [-R, 0]$, where $R > 0$ is thus the width of the past observation window, while the process D remains unobserved also in the past and $\Phi > 0$ on $[-R, 0]$. We notice that, if this is not the case, the robustness problem becomes trivial: in fact, if there exists R such that $\Phi(-R) = 0$, then the time window $[-R, 0]$ is sufficient to know D, since $\varphi \equiv 0$ on $(-\infty, R)$.

It turns out that also in this framework, as in [3], we can make the uncertainty on D decay with respect to the width R of the observation window. Again, we represent this uncertainty by defining the process $\widetilde{D}$, starting from the misspecified condition $\widetilde{D}_{-R}$ and following the dynamics

$$d\widetilde{D}_t = -\frac{\varphi(t)}{\Phi(t)}\widetilde{D}_t\, dt + dZ_t, \quad t \in (-R, 0] \tag{17}$$

while the process D always follows the dynamics given by Equation (7). Note that this time, since we can observe Z in the interval $[-R, 0]$, we have no uncertainty on this process.

The following lemma shows that, since the dynamics of $\widetilde{D}$ and D depend on the known values of Z, the difference between D_0 and $\widetilde{D}_0$ decays with respect to the width R, as announced.

Lemma 3. *Suppose that $\widetilde{D}$ and D have the dynamics* (17) *and* (7)*, respectively, and that at time $-R$ their values are $\widetilde{D}_{-R}$ and D_{-R}, respectively. Then*

$$D_0 - \widetilde{D}_0 = \frac{\Phi(-R)}{\Phi(0)} \left(D_{-R} - \widetilde{D}_{-R} \right) \ .$$

Proof. It is sufficient to observe that, by Equations (7) and (17), we have

$$\mathrm{d} \left(D_t - \widetilde{D}_t \right) = -\frac{\varphi(s)}{\Phi(s)} \left(D_t - \widetilde{D}_t \right) \mathrm{d}t \qquad \forall t \in (-R, 0] \ ,$$

so

$$D_0 - \widetilde{D}_0 = \mathrm{e}^{-\int_{-R}^{0} \frac{\varphi(s)}{\Phi(s)} \mathrm{d}s} (D_{-R} - \widetilde{D}_{-R}) = \frac{\Phi(-R)}{\Phi(0)} \left(D_{-R} - \widetilde{D}_{-R} \right) \ .$$

□

Now we are in the position of solving the following problem: for a given $\varepsilon > 0$ we want to find a minimum observation time R_0 such that the error when pricing a contingent claim h is less than ε. First we present a result on European claims, possibly path-dependent, which are Lipschitz with respect to the log-return Z.

Corollary 1. *Suppose that σ and σ^2 admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively.*

1. *Let $h : C^0[0,T] \to \mathbb{R}$ be the payoff of a claim such that the function $C^0[0,T] \to \mathbb{R} : f \mapsto h(\mathrm{e}^f)$ is globally Lipschitz with constant J. Then, for each $\varepsilon > 0$, if R is such that*

$$\Phi(-R) < \frac{\Phi(0)\varepsilon}{J \left(5 + \bar{\lambda}T\right) \mathrm{e}^{\left(\frac{L_1^2}{2} + \frac{L_2}{2} - \underline{\lambda}\right)^+ T} \mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right]} \tag{18}$$

   then one has

$$\left|\mathbb{E}\left[\mathrm{e}^{-rT} h\left(S(\cdot)\right)\right] - \mathbb{E}[\mathrm{e}^{-rT} h(\widetilde{S}(\cdot))]\right| < \varepsilon \ . \tag{19}$$

2. *Let $h : \mathbb{R} \to \mathbb{R}$ be the payoff of a European claim such that the function $\mathbb{R} \to \mathbb{R} : x \mapsto h(\mathrm{e}^x)$ is globally Lipschitz with constant J. In order for (19) to hold, it is sufficent that for each $\varepsilon > 0$,*

$$\Phi(-R) < \frac{\Phi(0)\varepsilon}{J \left(3 + \bar{\lambda}T\right) \mathrm{e}^{\left(\frac{L_2}{2} - \underline{\lambda}\right)^+ T} \mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right]} \tag{20}$$

Proof. For point 1., assuming (18) to be true, we have also

$$J \left(5 + \bar{\lambda}T\right) \mathrm{e}^{\left(\frac{L_1^2}{2} + \frac{L_2}{2} - \underline{\lambda}\right)^+ T} \frac{\Phi(-R)}{\Phi(0)} \mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right] < \varepsilon \ .$$

By Theorem 4 and Lemma 3, we finally conclude

$$\left|\mathbb{E}\left[\mathrm{e}^{-rT} h\left(S(\cdot)\right)\right] - \mathbb{E}[\mathrm{e}^{-rT} h(\widetilde{S}(\cdot))]\right| \leq J \left(5 + \bar{\lambda}T\right) \mathrm{e}^{\left(\frac{L_1^2}{2} + \frac{L_2}{2} - \underline{\lambda}\right)^+ T} \frac{\Phi(-R)}{\Phi(0)} \mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right] \ .$$

For point 2., the proof is analogous to that given above. □

The next result characterises the minimum observation time R such that the error when pricing a contingent claim h is less than ε when h is globally Lipschitz with respect to the natural variable S: again the result are for European claims which may also be path-dependent.

Corollary 2. *Suppose $d \mapsto \sigma(d)$ and $d \mapsto \sigma^2(d)$ admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively, and that $|\sigma|$ is bounded by M.*

1. *Let $h : C^0[0,T] \to \mathbb{R}$ be the payoff of a path-dependent claim having Lipschitz constant J. Then, for each $\varepsilon > 0$, if R is such that*

$$\Phi(-R) < \frac{\Phi(0)\varepsilon}{Jse^{\frac{M^2+\beta}{2}T}\left(L_2 T + 4L_1\sqrt{T}\right)\mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right]} \tag{21}$$

   then estimate (19) again holds.

2. *Let $h : C^0[0,T] \to \mathbb{R}$ be the payoff of a European claim having Lipschitz constant J. Then, for each $\varepsilon > 0$, if*

$$\Phi(-R) < \frac{\Phi(0)\varepsilon}{Jse^{\frac{M^2+\beta}{2}T}\left(\frac{L_2}{2} T + L_1\sqrt{T}\right)\mathbb{E}\left[\left|D_{-R} - \widetilde{D}_{-R}\right|\right]} \tag{22}$$

   then (19) holds.

Proof. Analogous to the previous proof, this time invoking Theorem 7. □

6 Stationarity: back to the Hobson-Rogers model

So far, we have seen that the problem of estimating the pricing error when we misspecify the offset function $\widetilde{D}$ reduces to the knowledge of $\mathbb{E}[|D_{-R} - \widetilde{D}_{-R}|]$, which in general is not allowed since we do not know the initial distribution of D_{-R}, even if we can decide the value $\widetilde{D}_{-R}$: this is a situation analogous to [3]. In that paper, a key step in obtaining a plausible value for this second moment was to assume the stationarity of the offset process D.

In this situation, a natural requirement for stationarity would be for M to be represented as

$$M_t = \int_{-\infty}^{0} \xi(u) Z_{t+u}\, du \tag{23}$$

for a suitable ξ such that $\int_{-\infty}^{0} \xi(u)\, du = 1$, so that

$$D_t = \int_{-\infty}^{0} \xi(u)(Z_t - Z_{t+u})\, du,$$

where in general we can only say that

$$M_t = \int_{-\infty}^{0} \frac{\varphi(t+u)}{\Phi(t)} Z_{t+u}\, du \quad \text{and} \quad D_t = \int_{-\infty}^{0} \frac{\varphi(t+u)}{\Phi(t)} (Z_t - Z_{t+u})\, du$$

This kind of stationarity would mean that the state variables M and D are obtained from the past prices of S via a rule which does not change over time, *i.e.* truly depends on $(S_u)_{u\in(-\infty,t]}$ only with respect to the values of the process S and not with respect to the time t.

In order to obtain representations as in Equation (23), we must impose that

$$\xi(u) = \frac{\varphi(t+u)}{\Phi(t)} \qquad \forall u \in (-\infty, 0], t \in \mathbb{R}$$

By taking $u = 0$, we obtain $\Phi(t)\xi(0) = \varphi(t) = \Phi'(t)$, which is a differential equation in Φ which has global solutions $\Phi(t) = K\mathrm{e}^{\xi(0)t}$. By letting $\lambda := \xi(0)$, we have

$$\xi(u) = \frac{\Phi'(t+u)}{\Phi(t)} = \frac{\lambda K \mathrm{e}^{\lambda(t+u)}}{K\mathrm{e}^{\lambda t}} = \lambda \mathrm{e}^{\lambda u}$$

This means that imposing the stationarity condition (23) is equivalent to requiring that $\Phi(t) = K\mathrm{e}^{\lambda t}$, which implies the original Hobson-Rogers dynamics for D. Also, we have that

$$\frac{\varphi(t)}{\Phi(t)} \equiv \lambda \quad \forall t \in \mathbb{R}$$

By plugging this result into Equations (7–8) we can see that the natural stationarity requirement (23) is equivalent to working with the Hobson-Rogers model, where D satisfies the SDE

$$dD_t = -\left(\lambda D_t + \frac{1}{2}\sigma^2(D_t)\right) dt + \sigma(D_t)\, dW_t \tag{24}$$

We now take from [3] a result on the stationarity of D. Define

$$G(x) = -\int_{x_0}^{x} \left(\frac{2\lambda u}{\sigma^2(u)} + 1\right) \mathrm{d}u$$

Theorem 8. *If*

$$\int_{-\infty}^{\infty} \frac{e^{G(x)}}{\sigma^2(x)}\, dx < \infty \quad \textit{and} \quad \int_{-\infty}^{0} \frac{e^{-G(x)}}{\sigma^2(x)}\, dx = \int_{0}^{\infty} \frac{e^{-G(x)}}{\sigma^2(x)}\, dx = +\infty,$$

then Equation (24) admits a unique stationary solution, and the associated invariant measure has a density f with respect to the Lebesgue measure given by the unique unitary integral solution of the equation

$$\left[\left(\lambda x + \frac{1}{2}\sigma^2(x)\right) f(x)\right]' + \frac{1}{2}\left(\sigma^2(x) f(x)\right)'' = 0 \ . \tag{25}$$

The above theorem leads to an explicit espression for the density:

$$f(x) = C\frac{\mathrm{e}^{G(x)}}{\sigma^2(x)} \tag{26}$$

where C is the suitable normalising constant. This means that it is possible to replace $\mathbb{E}[|D_{-R} - \widetilde{D}_{-R}|]$ with

$$V_1 := \int_{\mathbb{R}} |x - m_P| f(x)\, dx \tag{27}$$

where f is the density given above and we choose $\widetilde{D}_{-R} := m_P$ as the quantile of order $1/2$ of the invariant measure, which minimizes the function $y \to \int_{\mathbb{R}} |x - y| f(x)\, dx$. Notice that this is different from [3], where the authors chose $\widetilde{D}_{-R}$ as the expectation of the invariant measure, which was the minimizer there since their estimates were based on second order moments.

We can now restate Corollaries 1 and 2 in this new framework, by also noticing that, since Φ is now invertible, there will be a minimum R_0 such that Equations (18,20,21,22) hold. First we analyse the case when we have a log-Lipschitz contingent claim as in Section 3.

Corollary 3. *Suppose that σ and σ^2 admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively, and let $\Phi(t) := K\mathrm{e}^{\lambda t}$ and V_1 be defined as in Equation (27).*

1. *If $h : C^0[0,T] \to \mathbb{R}$ is the payoff of a path-dependent claim such that the function $C^0[0,T] \to \mathbb{R} : f \mapsto h(\mathrm{e}^f)$ is globally Lipschitz with constant J, and $R > R_0$, where*
$$R_0 := \left(\frac{L_1^2}{2\lambda} + \frac{L_2}{2\lambda} - 1\right)^+ T + \frac{1}{\lambda} \log \frac{J\,(5 + \lambda T)\, V_1}{\varepsilon} \tag{28}$$
then (19) holds.

2. *If $h : \mathbb{R} \to \mathbb{R}$ is the payoff of a European claim such that the function $\mathbb{R} \to \mathbb{R} : x \mapsto h(\mathrm{e}^x)$ is globally Lipschitz with constant J, then in order for (19) to hold it is sufficient that $R > R_0$, where now*
$$R_0 := \left(\frac{L_2}{2\lambda} - 1\right)^+ T + \frac{1}{\lambda} \log \frac{J\,(3 + \lambda T)\, V_1}{\varepsilon} \tag{29}$$

Then we analyse the case when we have a contingent claim which is globally Lipschitz with respect to the variable S, as in Section 4.

Corollary 4. *Suppose that $d \mapsto \sigma(d)$ and $d \mapsto \sigma^2(d)$ admit locally Lipschitz first partial derivatives, bounded by L_1 and L_2 respectively, and that $|\sigma|$ is bounded by M. Let $\Phi(t) := K\mathrm{e}^{\lambda t}$ and V_1 be defined in Equation (27).*

1. *If $h : C^0[0,T] \to \mathbb{R}$ is the payoff of a path-dependent claim having Lipschitz constant J and $R > R_0$, where*
$$R_0 := \frac{M^2 + \beta}{2\lambda} T + \frac{1}{\lambda} \log \frac{Js\left(L_2 T + 4 L_1 \sqrt{T}\right) V_1}{\varepsilon} \tag{30}$$
then (19) holds.

2. *If $h : C^0[0,T] \to \mathbb{R}$ is the payoff of a European claim having Lipschitz constant J, then in order for (19) to hold it is sufficient that $R > R_0$, where*
$$R_0 := \frac{M^2 + \beta}{2\lambda} T + \frac{1}{\lambda} \log \frac{Js\left(\frac{L_2}{2} T + L_1 \sqrt{T}\right) V_1}{\varepsilon} \tag{31}$$

We are now going to consider two determinations of σ that satisfy our assumptions, thereby calculating explicitly the density f and then the width R of the past window: this will be done by comparing the old robustness results from [3] with the ones obtained in this paper using the same numerical examples presented in [3].

6.1 The case $\sigma(D) = \min\left\{\sqrt{a+bD^2}, N\right\}$

Suppose that

$$\sigma(D) = \min\left\{\sqrt{a+bD^2}, N\right\} ,$$

where $a > 0$, $b > 0$ and $N > 0$ are constants, with $a < N^2$. We know from [3] that the unique invariant measure for the process D has a density of the form (26), where G is given by

$$G(x) = \int_{x_0}^{x} \frac{-\sigma^2(u) - 2\lambda u}{\sigma^2(u)}\,\mathrm{d}u = -(x - x_0) - \int_{x_0}^{x} \frac{2\lambda u}{\sigma^2(u)}\,\mathrm{d}u .$$

After some computations (see [3]), we obtain the formula

$$f(x) = \begin{cases} K_1 \mathrm{e}^{-\frac{\lambda(N^2-a)}{bN^2} - \frac{N^2}{4\lambda}} N^{\frac{2\lambda}{b}} \mathrm{e}^{-x} (a+bx^2)^{-\frac{\lambda}{b}-1} & \text{if } |x| \le \sqrt{\frac{N^2-a}{b}} \\ \frac{K_1}{N^2} \mathrm{e}^{-\frac{\lambda}{N^2}\left(x+\frac{N^2}{2\lambda}\right)^2} & \text{if } |x| \ge \sqrt{\frac{N^2-a}{b}} \end{cases}$$

where K_1 is a convenient constant.

Example 1. As in [10] and [3], we take

$$a = 0.04, \quad b = 0.2, \quad \lambda = 1, \quad N = 1$$

so we have

$$L_1 = \sup_{x\in\mathbb{R}} \left|\frac{\partial \sigma}{\partial x}\right| = \frac{\sqrt{b(N^2-a)}}{N} = 0.438178$$

and

$$L_2 = \sup_{x\in\mathbb{R}} \left|\frac{\partial \sigma^2}{\partial x}\right| = 2\sqrt{b(N^2-a)} = 0.876356$$

and we obtain

$$V_1 = \mathbb{E}[|P - m_P|] = 0.116144 .$$

We want to find R such that the pricing error is less than $\varepsilon = 10^{-2}$, both for a path-dependent contingent claim as well as for a European one, both with Lipschitz constant $J = 1$. By taking different maturities, we find the results in Table 1: we use R_{HV} to denote the observation window obtained with the original estimates of [3], and R for the observation window obtained with the estimates (18) and (20) of Corollary 1.

We can see a huge improvement of the new results obtained here over those in [3], which is evident especially for longer maturities: in fact, while in order to price a 5-years contingent claim with an error of less than $\varepsilon = 10^{-2}$ with the old estimates from [3] one needed an observation window of more than a century, with the results of this paper one knows that the necessary time window is really less than 5 years long.

There is an analogous situation with respect to simple European contingent claims: in fact, here too we can see a huge improvement in longer maturities: for the same maturity as above (5 years), and once again we pass from an observation window of about a century to about 4 and a half years.

	path-dependent		European	
T	R_{HV}	R	R_{HV}	R
0.25	3.971	4.110	3.611	3.631
0.5	5.157	4.157	4.438	3.705
1.0	8.943	4.244	7.504	3.839
2.0	22.167	4.398	19.288	4.062
3.0	42.927	4.532	38.608	4.244
4.0	71.223	4.649	65.464	4.398
5.0	107.055	4.755	99.856	4.532

Table 1: time to wait (in years) for a precision $\varepsilon = 0.01$: with R the estimate with the current method, with R_{HV} with the one in [3]

6.2 The case $\sigma^2(D) = \frac{a+bD^2}{c+dD^2}$

Let us now suppose that σ is of the form

$$\sigma^2(D) = \frac{a + bD^2}{c + dD^2} \ ,$$

where a, b, c and d are strictly positive constants. Again in this case we know from [3] that there exists a unique invariant measure for the process D, having a density of the form (26), where

$$G(x) = -(x - x_0) - 2\lambda \int_{x_0}^{x} \frac{(c + du^2)u}{a + bu^2} \, \mathrm{d}u \ .$$

By an easy computation (see [3]) one concludes that the density is given by

$$\begin{aligned} f(x) = C\frac{\mathrm{e}^{G(x)}}{\sigma^2(x)} &= C\frac{\mathrm{e}^{-x}(bx^2 + a)^{-\frac{\lambda}{b^2}(bc-ad)}\mathrm{e}^{-\frac{\lambda d}{b^2}(bx^2+a)+c_1}}{\frac{a+bx^2}{c+dx^2}} \\ &= K\frac{\mathrm{e}^{-\frac{\lambda d}{b}\left(x+\frac{b}{2\lambda d}\right)^2}(bx^2 + a)^{-\frac{\lambda}{b^2}(bc-ad)-1}}{c + dx^2} \end{aligned}$$

where K is a convenient constant.

Example 2. As in [10] and [3], we take

$$a = 0.452, \quad b = 3.012, \quad c = 1.0, \quad d' = 0.261, \quad \lambda = 1.02$$

so we have

$$L_1 = \sup_{x \in \mathbb{R}} \left| \frac{\partial \sigma}{\partial x} \right| = 1.22302$$

and

$$L_2 = \sup_{x \in \mathbb{R}} \left| \frac{\partial \sigma^2}{\partial x} \right| = \frac{9|bc - ad'|\sqrt{\frac{c}{3d'}}}{8c^2} = 3.67938$$

then we have

$$V_1 = \mathbb{E}[|P - m_P|] = 0.459656 \ .$$

We want to find R such that the pricing error is less than $\varepsilon = 10^{-2}$, both for a path-dependent contingent claim as well as for a European claim, both with

Lipschitz constant $J = 1$. By taking different maturities, we find the results in Table 2: as before we indicate with R_{HV} the observation window obtained with the original estimates of [3], and with R, the observation window obtained with the estimates (18) and (20) of Corollary 1.

	path-dependent		European	
T	R_{HV}	R	R_{HV}	R
0.25	9.500	5.744	6.750	5.091
0.5	16.228	6.175	10.729	5.366
1.0	35.810	7.030	24.811	5.901
2.0	99.471	8.720	77.475	6.926
3.0	195.798	10.390	162.803	7.911
4.0	324.789	12.044	280.796	8.867
5.0	486.445	13.685	431.453	9.802

Table 2: time to wait (in years) for a precision $\varepsilon = 0.01$: with R the estimate with the current method, with R_{HV} with the one in [3]

Also in this case we can see a huge improvement in the new results obtained here, which is again evident especially for longer maturities: in fact, while in order to price a 5-years contingent claim with an error of less than $\varepsilon = 10^{-2}$ with the old estimates from [3] one needed the astonishing observation window of almost five centuries, with the results of this paper one knows that the necessary time window is really less than 14 years long.

When analysing the case of simple European claims with this volatility specification, the situation is analogous. In fact, here too we can see a huge improvement in longer maturities: for the same maturity as above (5 years), we pass from an observation window of about four centuries to about 10 years.

7 Calibration

We calibrate the model, using both the volatility specifications of Sections 6.1 and 6.2, to market data of January 7, 2009 (which corresponds to $t = 0$) of the European index EUROSTOXX50 and of the Spanish index IBEX. For the calibration we used European calls and puts with strike prices presenting significant trade volumes and maturities up to 1 year. As a proxy for r we used the value of the 1-year EURIBOR rate, which was at $r = 2.959\%$, and both the indexes presented an implied dividend, which was taken into account in the calculations. The numerical procedure used for the calibration is the same used in [7, 9] (a mean square calibration using a Kolmogorov finite difference scheme for the theoretical prices, see [9] for details).

More in detail, we calibrate the model using both the volatility specification

$$\sigma_1(D) = \min\left\{\sqrt{a + bD^2}, N\right\} , \tag{32}$$

of Section 6.1, for which we have to calibrate the parameters N, a, b and λ, as well as the volatility specification

$$\sigma_2(D) = \sqrt{\frac{a + bD^2}{c + dD^2}} , \tag{33}$$

of Section 6.2, for which we must calibrate the parameters a, b, c, d and λ. This means that, for each choice of the volatility σ_i, $i = 1, 2$, here the parameters to calibrate are the ones of each volatility specification and λ so at every iteration we have to recalculate the initial value of the offset function that depends on λ and on the prices of S in the observation window $[-R, 0]$. Like many models, here the calibration is not perfect in the sense that, after the calibration is done with respect to a volatility shape σ_i, $i = 1, 2$, for a fixed strike K, maturity T and type of contract (call or put) we have a theoretical price f_i given by the model and an observed price $\hat{f}$, with error given by

$$\varepsilon_{tot}^{i} := \frac{f_i - \hat{f}}{S_0} \tag{34}$$

i.e. scaled with respect to the underlying S_0. In principle another choice is possible, see Remark 3 below.

Provided the Hobson-Rogers model is the right one, this error can still be different from zero due to two facts. The first is that, despite having found the optimal parameters for σ_i, this volatility shape could simply not be the right one. The second is the very scope of this paper, i.e. we have a finite observation horizon $R < +\infty$. The analysis of the error due to the first reason would deserve another paper by itself, so we can say that it goes beyond the scope of the present one. Conversely, here we concentrate on the error due to the second reason. In particular, in Corollaries 1–4 we found sufficient conditions on R for the (absolute) pricing error to be less than a given ε_{hor}, provided that there are not other sources of errors. If these are present (for example, if the volatility is not the right one), it is possible that $\varepsilon_{tot} S_0 > \varepsilon_{hor}$ (recall that ε_{tot} is the error scaled with respect to S_0, while ε_{hor} is an absolute error). However, if the converse is true, this could mean that all the pricing error comes from the fact that the observation window is too short. The discussion above, though not rigorous, seems a valid argument to say that the two pricing errors "tot" and "hor" should have the same order of magnitude, once we choose the scale (absolute and/or with respect to S_0).

Remark 3. *In principle, one could just as well use the relative error*

$$\bar{\varepsilon}_{tot}^{i} := \frac{f_i - \hat{f}}{\hat{f}} \tag{35}$$

as a measurement of the error. We, however, feel that in this case the most natural error to use is the one given in Equation (34), as that quantity can be easily incorporated in Equations (30) and (31), which give the robustness estimates that will be used here. In fact, it is sufficient to let $\varepsilon := \varepsilon_{tot}^{i} S_0$ in those equations to get the minimum R_0, also obtaining a simplification for $s = S_0$. Conversely, when one uses $\bar{\varepsilon}_{tot}^{i}$ the financial interpretation is less immediate. In any case, for the sake of completeness, we also indicate $\bar{\varepsilon}_{tot}^{i}$ in the calibration Tables 3 and 4.

Once that this is settled, we have to check that the depth R of the observation window that we used is sufficient to achieve this. In particular, in order to calibrate we first have to fix R, then calibrate for the volatility parameters and for λ, then calculate the relative errors ε_{tot} for each different claim. Once all this is done, we can choose ε_{hor} in the light of the previous discussion and see if the depth R that we used is greater or less than the minimum R_0 given by Corollary 4. In the following two examples we fix $R = 10$ years (for availability of electronic data) and then validate if this was enough to have ε_{hor} comparable with ε_{tot} or not for the derivatives used for the calibration (which in both cases have maturities $T < 1$ year). In that case we are also able to calculate the minimal observation window R_0 for a generic derivative having an arbitrary maturity T.

7.1 Underlying: EUROSTOXX50

The underlying initial value of the index was $S_0 = 2560$, with an implied annualized dividend of $q = 4.6\%$. In Table 3 we use $\hat{f}$ to denote the observed prices of European calls and puts used for calibration, f_1 for the price after we calibrate the model using the volatility specification of Equation (32), and f_2 for the price after we calibrate the model using the volatility specification of Equation (33). For both the volatility specifications we compute the total scaled error, computed as in Equation (34). In particular, in Table 3, in the first column we indicate the type of contract (call or put), in the second one the strike price, in the third one the maturity (5, 8 or 11 months); in the fourth the observed prices. In the fifth and sixth, respectively, the calibrated price and relative errors (see Equations (34) and (35)) using the volatility specification σ_1, while in the seventh and eighth, respectively, the calibrated price and relative errors using the volatility specification σ_2.

C/P	K	T	$\hat{f}$	f_1	ε^1_{tot}	$\bar{\varepsilon}^1_{tot}$	f_2	ε^2_{tot}	$\bar{\varepsilon}^2_{tot}$
call	2300	5/12	371.0	358.8	−0.0047	−0.032	359.0	−0.0047	−0.032
call	2600	5/12	190.8	205.9	+0.0059	+0.079	209.5	+0.0073	+0.098
call	2400	8/12	355.1	354.2	−0.0003	−0.002	351.8	−0.0013	−0.009
call	2600	8/12	243.1	269.4	+0.0103	+0.108	268.3	+0.0099	+0.103
call	2400	11/12	387.3	389.5	+0.0009	+0.005	388.3	+0.0004	+0.002
call	2600	11/12	278.2	307.2	+0.0113	+0.104	307.1	+0.0113	+0.103
put	2300	5/12	156.8	117.9	−0.0152	−0.248	118.0	−0.0151	−0.247
put	2600	5/12	275.0	261.4	−0.0053	−0.049	265.0	−0.0039	−0.036
put	2400	8/12	240.0	223.9	−0.0063	−0.067	221.4	−0.0073	−0.077
put	2600	8/12	324.9	334.6	+0.0038	+0.029	333.6	+0.0034	+0.026
put	2400	11/12	279.3	270.0	−0.0036	−0.033	268.9	−0.0041	−0.037
put	2600	11/12	366.6	382.0	+0.0060	+0.042	381.9	+0.0060	+0.041

Table 3: Observed and calibrated prices for European calls and puts option on EUROSTOXX50 for the day January 7, 2009. In the first column we indicate the type of contract (call or put). Calibrated parameters values for σ_1: $N = 0.9363$, $a = 0.1244$, $b = 0.4096$, $\lambda = 0.75$, offset function $D_0 = -0.2975$. Calibrated parameter values for σ_2: $a = 0.3097$, $b = 2.0297$, $c = 2.6049$, $d = 1.0276$, $\lambda = 0.5582$, offset function $D_0 = -0.2815$.

From Table 3 we notice that with the two volatility specifications σ_1 and σ_2 the expected prices give values close to each other across the various strikes and maturities, and the maximum scaled error given by Equation (34) is around 1.5% of the underlying for both σ_1 and σ_2, while the maximum relative error given by Equation (35) is around 10%, with a single exception of 24.8%. Furthermore, after the calibration, we can plot the two volatility specifications in Figure 1, which appear close as functions of the offset D_t, at least for values of D_t which are not too extreme.

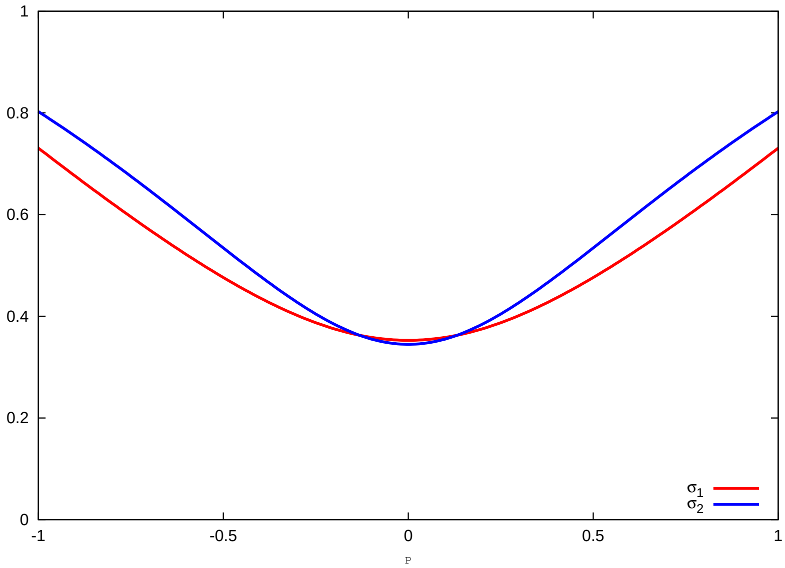

Figure 1: Calibrated volatility shapes with respect to the offset function D for the EUROSTOXX index.

We can now calculate the minimum window width R_0 such that Equations (30) and (31) are verified. We first assume $J = 1$ (as is for vanilla calls and puts) and allow for a maximum error $\varepsilon_{hor} = 1\% S_0 = 0.01 S_0$, which is of the same order of magnitude of the relative errors ε_{tot} in Table 3. We also denote by R_1 the observation width when using the volatility specification σ_1, and by R_2 the observation width when using the volatility specification σ_2, this both for path-dependent claims and for European claims. We obtain the following table for some significant maturities T.

	path-dependent		European	
T	R_1	R_2	R_1	R_2
0.25	5.0259	7.1769	3.4091	4.9319
0.5	5.7349	8.3694	4.1827	6.1978
1.0	6.6200	10.0640	5.1406	7.9793
2.0	7.8319	12.7367	6.4306	10.7500
3.0	8.8010	15.0999	7.4460	13.1740
4.0	9.6665	17.3317	8.3441	15.4499
5.0	10.4737	19.4898	9.1761	17.6422

From this table, first of all we can validate the calibration of Table 3, because for $\varepsilon_{hor} = 0.01S_0$ and $T \leq 1$ year for vanilla calls and puts we have $R_0 \leq 7.97$ when using σ_2 and $R_0 \leq 5.14$ when using σ_1. We can then use the other numbers of the table for a generic claim of arbitrary maturity up to 5 years: from the table it is quite evident that, if one wants to use the minimal horizon with a given precision, in this case it is better to use the volatility specification σ_1.

7.2 Underlying: IBEX

The underlying initial value was $S_0 = 9620$, with an implied annualized dividend of $q = 5.7\%$. In Table 4 we again use $\hat{f}$ to denote the observed prices of European calls and puts used for calibration, f_1 for the price after the calibration with the volatility specification (32) of Section 6.1, and f_2 for the price after the calibration with the volatility specification (33) of Section 6.2. In particular, again in the first column we indicate the type of contract (call or put); in the second one the strike price, in the third one the maturity (5, 8 or 11 months); in the fourth the observed prices; in the fifth and sixth, respectively, the calibrated price and relative errors using the volatility specification σ_1, while in the seventh and eighth, respectively, the calibrated price and relative errors using the volatility specification σ_2.

C/P	K	T	$\hat{f}$	f_1	ε^1_{tot}	$\bar{\varepsilon}^1_{tot}$	f_2	ε^2_{tot}	$\bar{\varepsilon}^2_{tot}$
call	9600	5/12	782	729.8	−0.0054	−0.066	756.6	−0.0026	−0.032
call	9700	5/12	728	693.6	−0.0036	−0.047	720.4	−0.0008	−0.010
call	9600	8/12	910	924.4	+0.0020	+0.015	938.4	+0.0030	+0.031
call	9700	8/12	909	893.8	−0.0016	−0.016	902.2	−0.0007	−0.007
call	9600	11/12	1034	1043.8	+0.0010	+0.009	1060.5	+0.0028	+0.025
call	9700	11/12	983	1007.9	+0.0026	+0.025	1024.0	+0.0043	+0.041
put	9600	5/12	897	815.5	−0.0085	−0.090	842.4	−0.0057	−0.060
put	9700	5/12	943	876.3	−0.0069	−0.070	903.1	−0.0041	−0.042
put	9600	8/12	1103	1077.0	−0.0027	−0.023	1086.0	−0.0018	−0.015
put	9700	8/12	1187	1138.2	−0.0051	−0.041	1146.7	−0.0042	−0.033
put	9600	11/12	1292	1252.3	−0.0041	−0.030	1269.0	−0.0024	−0.017
put	9700	11/12	1339	1313.2	−0.0027	−0.019	1329.3	−0.0010	−0.007

Table 4: Observed and calibrated prices for European calls and puts option on IBEX for the day January 7, 2009. In the first column we indicate the type of contract (call or put). Calibrated parameters values for σ_1: $N = 0.9668$, $a = 0.1011$, $b = 0.3978$, $\lambda = 0.6597$, offset function $D_0 = -0.1552$. Calibrated parameter values for σ_2: $a = 0.3873$, $b = 1.7415$, $c = 3.6399$, $d = 3.9626$, $\lambda = 0.6$, offset function $D_0 = -0.1389$.

From Table 4 we notice that with the two volatility specifications σ_1 and σ_2 the expected prices give values close to each other across the various strikes and maturities, and the maximum scaled error given by Equation (34) is always less than 1% of the underlying for both σ_1 and σ_2, while the maximum relative error given by Equation (35) is always less than 9%. Furthermore, after calibration, we can plot the two volatility specifications in Figure 1, which appear close as functions of the offset D_t.

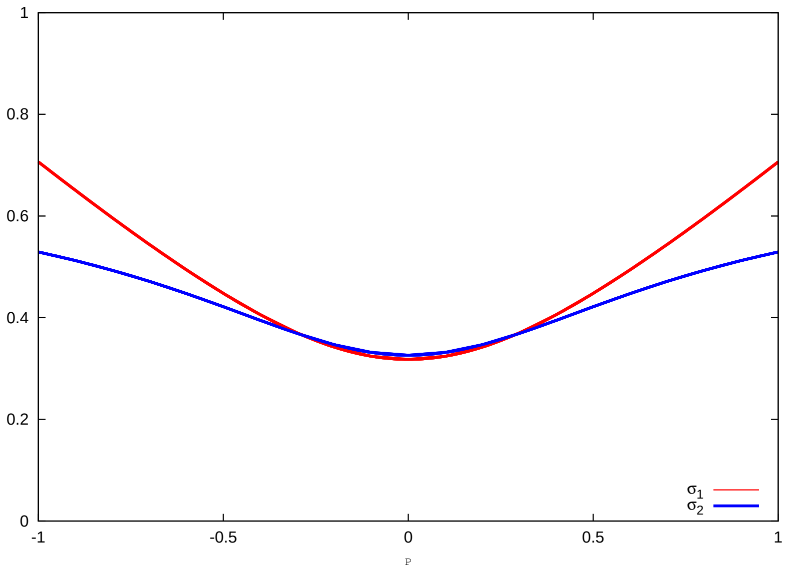

Figure 2: Calibrated volatility shapes with respect to the offset function D for the IBEX index.

We can now again calculate the minimum window width R_0 such that Equations (30) and (31) are verified. We again assume $J = 1$ and allow for a maximum error $\varepsilon_{hor} = 1\%S_0 = 0.01S_0$, which is again of the same order of magnitude of the errors ε_{tot} of Table 4, and as before we use R_1 to denote the observation width when using the volatility specification σ_1 and R_2 for the observation width when using the volatility specification σ_2. This notation is for both path-dependent claims and European claims. This time, for the same significant values of T as in Table 7.1, we obtain the following table.

	path-dependent		European	
T	R_1	R_2	R_1	R_2
0.25	5.7466	4.3809	3.9148	2.2221
0.5	6.6020	5.1368	4.8446	3.0294
1.0	7.7045	6.0290	6.0307	3.9859
2.0	9.2721	7.1775	7.6872	5.2114
3.0	10.5620	8.0541	9.0298	6.1385
4.0	11.7335	8.8163	10.2382	6.9385
5.0	12.8383	9.5145	11.3711	7.6669

From this table, first of all we can validate the calibration of Table 4, because for $\varepsilon_{hor} = 0.01S_0$ and $T \leq 1$ year for vanilla calls and puts we have $R_0 \leq 3.98$ when using σ_2 and $R_0 \leq 6.03$ when using σ_1. We can then use the other numbers of the table for a generic claim of arbitrary maturity up to 5 years: from the table it is quite evident that, if one wants to use the minimal horizon with a given precision, this time it is better to use the volatility specification σ_2.

A Auxiliary results

The next result is used when differentiating a Lebesgue integral; though in the proof we essentially use standard limit theorems, we provide an explicit proof as we could not find this result in standard textbooks.

Proposition 1. *Let $f : \mathbb{R}\times[0,t] \to \mathbb{R}$ be a measurable function, with first partial derivative with respect to the first variable $\partial_1 f \geq 0$. Suppose that for every $x \in \mathbb{R}$, $f(x,\cdot) \in L^1[0,t]$, $\partial_1 f(x,\cdot) \in L^1[0,t]$, for each $s \in [0,t]$ $\partial_1 f(\cdot,s) \in L^1_{loc}(\mathbb{R})$ and $x \mapsto \int_0^t f(x,s)\,ds$ is differentiable and belongs to $L^1_{loc}(\mathbb{R})$. Then, for almost every $x \in \mathbb{R}$, we have*

$$\int_0^t \partial_1 f(x,s)\,ds = \left(\int_0^t f(\cdot,s)\,ds\right)'(x)\ .$$

Proof. Since f is measurable, the same is true for $\partial_1 f$. The positivity of $\partial_1 f$ permits us to write

$$\begin{aligned}
&\int_0^y \int_0^t \partial_1 f(x,s)\,\mathrm{d}s\,\mathrm{d}x = \int_0^t \int_0^y \partial_1 f(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t (f(y,s) - f(0,s))\ \mathrm{d}s = \left(\int_0^t f(\cdot,s)\,\mathrm{d}s\right)(y) - \left(\int_0^t f(\cdot,s)\,\mathrm{d}s\right)(0) = \\
&= \int_0^y \left(\int_0^t f(\cdot,s)\,\mathrm{d}s\right)'(x)\,\mathrm{d}x
\end{aligned}$$

where the second and fourth equalities follow from [19, Theorem 8.21] and the hypothesis on $\partial_1 f(\cdot,s)$ and $\int_0^t f(\cdot,s)\,\mathrm{d}s$. This implies that

$$\int_I \int_0^t \partial_1 f(x,s)\,\mathrm{d}s\,\mathrm{d}x = \int_I \left(\int_0^t f(\cdot,s)\,\mathrm{d}s\right)'(x)\,\mathrm{d}x \tag{36}$$

for every I belonging to the ring of finite union of disjoint bounded intervals of $\mathbb{R}$. Identity (36) implies

$$\left(\int_0^t f(x,s)\,\mathrm{d}s\right)' \geq 0 \qquad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, an easy application of the monotone convergence theorem (for increasing sequences of sets), and of Lebesgue convergence theorem (for decreasing sequences of sets), leads to state equation (36) for each I belonging to the monotone class generated by finite unions of disjoint bounded intervals of $\mathbb{R}$, *i.e.* for I Borel, which finally implies

$$\int_0^t \partial_1 f(x,s)\,\mathrm{d}s = \left(\int_0^t f(\cdot,s)\,\mathrm{d}s\right)'(x)\ .$$

for almost every $x \in \mathbb{R}$. □

References

[1] Antonelli, Fabio, Prezioso, Valentina, "Rate of convergence of Monte Carlo simulations for the Hobson-Rogers model". Int. J. Theor. Appl. Finance **11 (8)** (2008), 889–904.

[2] Blaka Hallulli, V., T. Vargiolu, "Financial models with stochastic dependence on the past: a survey", in *Applied industrial mathematics in Italy*, editors M. Primicerio *et al.*, Series on Advances in Mathematics for Applied Sciences, 69, pp. 348–359, World Scientific, 2005.

[3] Blaka Hallulli, V., T. Vargiolu, "Robustness of the Hobson-Rogers model with respect to the offset function", in Proceedings of the Ascona '05 Seminar on Stochastic Analisys, Random Fields and Application, R. C. Dalang, M. Dozzi, F. Russo, editors, Birkhäuser, pp. 469–492, 2007.

[4] K. Burdzy, *Some path properties of iterated Brownian motion*, Seminar on Stochastic Processes, E. Cinlar (ed.) et al., Birkhäuser, Boston. (1993), 67–87

[5] Chiarella, C., Kwon, K. "A complete Markovian stochastic volatility model in the HJM framework". Asia-Pacific Financial Markets **7 (4)** (2000), 293–304

[6] Da Prato, G., *Introduction to differential stochastic equations*, 2^{nd} ed. Scuola Normale Superiore, Pisa, 1998.

[7] Di Francesco M., Pascucci A., "On the complete model with stochastic volatility by Hobson and Rogers". Proc. R. Soc. Lond. A Vol. 460 (2004), 3327–3338

[8] Figà-Talamanca, G., Guerra, M.L., "Fitting prices with a complete model", Journal of Banking and Finance, 30(1) (2006), 247–258

[9] Foschi, P., A. Pascucci, "Calibration of a path-dependent volatility model: empirical tests". Comput. Statist. Data Anal., Volume 53 (2009), 2219–2235

[10] Foschi, P., A. Pascucci, "Path dependent volatility", *Decisions in Economics and Finance*, 31 (1), 2008, pp. 1–20.

[11] Has'minskiĭ, R.Z., *Stochastic stability of differential equations*, Alphen den Rijn, Sijthoff & Noordhoff, 1980.

[12] Hobson, D.G., L.C.G. Rogers, "Complete models with stochastic volatility", *Mathematical Finance*, 8 (1), 1998, pp. 27–48.

[13] Hull, J.C., *Options, futures, and other derivatives*, 5^{th} ed., Prentice-Hall-International, 2002.

[14] Kawai, R., "Sensitivity analysis and density estimation for the Hobson-Rogers stochastic volatility model", *International Journal of Theoretical and Applied Finance*, 12 (3), 2009, pp. 283–295

[15] Lamberton, D., B. Lapeyre, *Introduction to stochastic calculus applied to finance*, London, Chapman & Hall, 1996.

[16] Øksendal, B., *Stochastic differential equations*, 5th ed., Springer, 2000.

[17] Platania, A., L.C.G. Rogers, "Putting the Hobson–Rogers model to the test", preprint, 2003. Available at: `http://www.statslab.cam.ac.uk/~chris/papers/AP_CR_HR.pdf`

[18] Protter, P.E., *Stochastic integration and differential equations*, 2nd ed., Springer, 2004.

[19] Rudin, W., *Real and complex analysis: international student edn.*, McGraw-Hill, 1970.

[20] Sekine, J., "Marginal distributions of some path-dependent stochastic volatility model". Statistics and Probability Letters 78 (2008), 1846–1850