Modeling and valuing make-up clauses in gas swing contracts

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Abstract

In the last ten years, thanks to the worldwide energy liberalization process, the birth of competitive gas markets and the recent financial crisis, traditional long term swing contracts in Europe have been supplemented in a significant way by make-up clauses which allow to postpone the withdrawal of gas to future years when it could be more profitable. This introduces more complexity in the pricing and optimal management of swing contracts. This paper is devoted to a proper quantitative modelization of a kind of make-up clause in a gas swing contract. More in detail, we succeed in building an algorithm to price and optimally manage the make-up gas allocation among the years and the gas taking in the swing sub-periods within the years: we prove that this problem has a quadratic complexity with respect to the number of years. The algorithm can be adapted to different instances of make-up clauses as well as to some forms of carry-forward clauses. Then, as an example, we show the algorithm at work on a 3-year contract and we present a sensitivity analysis of the price and of the make-up policy with respect to various parameters relative both to the price dynamics as well as to the swing contract. To the authors’ knowledge, this is the first time that such a quantitative treatment of make-up clauses appears in literature.

Keywords: swing option; price decoupling; make-up clause; dynamic programming; bang-bang controls.

JEL Classification: C61, C63, D81, Q49.

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1 Introduction to long term supply contracts in European gas market

Europe is among the largest consumer of natural gas in the world, mainly used for heating and power generation. During the last thirty years natural gas has gradually replaced almost everywhere fuel oil for heating purposes and is actually competing with coal as main fuel source for electric power generation. Hence, long term trend of natural gas demand has been historically upward sloping. The economic crisis of 2008 has strongly impacted this tendency: global gas demand fell sharply by 3% between 2008 and 2009. As reported in Table 1, the International Energy Agency (IEA) forecasts that OECD\(^1\) gas demand would recover slowly with consumption returning to the 2008 levels by 2012 or 2013, depending on the region.

<table>
<thead>
<tr>
<th></th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Europe</td>
<td>557</td>
<td>527</td>
<td>544</td>
<td>533</td>
<td>540</td>
<td>548</td>
</tr>
<tr>
<td>North America</td>
<td>814</td>
<td>800</td>
<td>804</td>
<td>803</td>
<td>820</td>
<td>835</td>
</tr>
<tr>
<td>Pacific</td>
<td>175</td>
<td>169</td>
<td>180</td>
<td>182</td>
<td>189</td>
<td>195</td>
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<tr>
<td>Total</td>
<td>1546</td>
<td>1496</td>
<td>1528</td>
<td>1518</td>
<td>1549</td>
<td>1578</td>
</tr>
</tbody>
</table>

Table 1: OECD Natural Gas Demand by Region in billion cubic meters per year. Datas for 2008-2009 are historical, for 2010 estimated, and for 2011-2013 forecasted \[18\].

Recent events concerning nuclear power generation, post Fukushima’s accident, are expected to provide new strength to the long term up-growing tendency of natural gas global demand. In fact, in the medium to long term, many countries are expected to reduce their nuclear ambitions and the fuel of choice to compensate for lower nuclear will reasonably be natural gas.

Despite its significant consumption, Europe, meant either as OECD or European Union (EU), has only a limited inner production compared to its consumption and the excess demand is covered by massive natural gas imports from producer countries like Russia and Algeria, as shown in Table 2 where the Natural Gas Imports for the EU-27\(^2\) countries is reported.

Natural gas imports are physically delivered via pipelines (Figure 1) or via LNG (Liquified Natural Gas) cargoes and were traditionally based on long term oil-linked swing contracts (10-30 years duration) in order to guarantee the security of supply of such an important energy commodity. In Europe long term gas contracts have been traditionally priced using oil-linked pricing formulas, while in the United States gas-to-

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\(^1\)Current membership of OECD: Australia, Austria, Belgium, Canada, Chile, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States.

\(^2\)EU-27: Austria, Belgium, Bulgaria, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden and the United Kingdom.
Figure 1: Existing and planned major European pipelines routes [23].

Table 2: EU-27 Natural Gas imports for year 2009 by country of origin [11].

<table>
<thead>
<tr>
<th>Country</th>
<th>Tj/y</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Russia</td>
<td>4'524'090.00</td>
<td>28.7%</td>
</tr>
<tr>
<td>Norway</td>
<td>4'055'038.00</td>
<td>25.7%</td>
</tr>
<tr>
<td>Algeria</td>
<td>1'868'376.00</td>
<td>11.9%</td>
</tr>
<tr>
<td>Netherlands</td>
<td>1'691'445.00</td>
<td>10.7%</td>
</tr>
<tr>
<td>UK</td>
<td>408'050.00</td>
<td>2.6%</td>
</tr>
<tr>
<td>Lybia</td>
<td>380'143.00</td>
<td>2.4%</td>
</tr>
<tr>
<td>Egypt</td>
<td>274'637.00</td>
<td>1.7%</td>
</tr>
<tr>
<td>Others</td>
<td>2'556'190.00</td>
<td>16.2%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>15'757'969.00</strong></td>
<td><strong>100.0%</strong></td>
</tr>
</tbody>
</table>

Table 3: Natural Gas Transactions by Pricing Mechanism in 2007 [26].

<table>
<thead>
<tr>
<th></th>
<th>Oil Linked</th>
<th>Gas Market Linked</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Europe</td>
<td>72%</td>
<td>22%</td>
<td>6%</td>
</tr>
<tr>
<td>North America</td>
<td>0%</td>
<td>99%</td>
<td>1%</td>
</tr>
<tr>
<td>Pacific</td>
<td>52%</td>
<td>16%</td>
<td>32%</td>
</tr>
</tbody>
</table>

Gas competition has historically determined most of natural gas wholesale transactions.

In the last ten years, thanks to the worldwide energy liberalization process and the birth of competitive gas markets, in almost all European countries these long term contracts have been supplemented by spot transactions (short term transactions) even if long term deals still represent the pillar of European gas system (see Table 3 and [20]).

The structure of long term gas agreements is pretty standardized in Europe. As already said, these long term contracts are swing (also known as take-or-pay, see [19] for details) in nature, with the peculiarity that the strike price typically depends upon a basket of crude and refined oil products, which is averaged through time in order to smooth undesired volatility effects; for more details we refer the interested reader to [1, Section 3.1]. Since oil products are traded in US dollars, oil related indexes are also expressed in US dollars, thus typical market risk factors perceived by European importers are represented both by USD/EUR exchange risk, gas cost $I_n$ and price differential between import cost in Euros and local market prices $P_n$, settled daily by gas market exchanges. We however emphasize that USD/EUR exchange rate volatility is comparatively low compared with typical spot gas price volatility (Figure 2(a)).

These long term contract structure and oil indexation have their origins in the early
European gas market of the 1970s. Since that time sources of gas have increased, making gas markets and infrastructure much denser and open to competition. From 2008 onwards this traditional market framework has significantly changed especially for what concern the oil-to-gas price relationship. Actually, the demand drop following the financial crisis with the subsequent economic recession associated to the significant increase in LNG (Liquefied Natural Gas) and unconventional gas supply sources flowing to Europe generated a consistent and pretty persistent oil decoupling of European gas market prices: since 2008, European gas markets are pricing systematically and significantly below indexes usually used for the strike price \( I \) above (see Figure 2(b)), so the spread \( P - I \) has become negative.

Obviously, this market phenomenon has determined a panic situation for all the owners of classical long term gas supply contracts. Significant losses have been faced at present by pipeline importers due to this kind of oil-to-gas price decoupling; moreover, the structural market change determined an increased sense of uncertainty about European gas market future development. Interested readers may refer to [20] for a detailed and updated analysis of oil-to-gas decoupling.

This new market scenario has induced many long term importers to engage a renegotiation process with their suppliers together with a more focused attention towards optimization and hedging possibilities which are naturally embedded in the current contracts. In fact, traditionally long term forward gas contracts are often equipped with some volumetric flexibilities in order to intertemporally manage gas demand fluctuations year by year. Among those, in this market situation a new and particular impor-
tance arose for the so-called make-up and carry forward clauses, which flank traditional constraints as minimum and maximum withdrawal quantity established for every contract year and every contract sub-period (day or month). Basically, these clauses allow the buyer of the contract to delay or anticipate respectively the withdrawal of gas from one year to another within the full respect of sub-period capacity constraints. In particular, the introduction of make-up clauses has become very important for European long term contracts holders: in fact, in the recent oil-to-gas price decoupling situation, contracts holders were induced to delay as much as possible the gas delivery for the sake of loss minimization. With a make-up clause contract holders can effectively postpone the delivery of gas when it is too expensive with respect to market prices, hoping that future gas prices will rise up and the exercise of the contract rights becomes again profitable.

Furthermore, gas and oil markets are extremely volatile (see, for example, the volatility of gas market TTF (Title Transfer Facility, Netherlands gas hub) in Figure 2(a)) and consequently contracts optimization and hedging has to be performed dynamically through time in order to protect contract’s value or at least contain financial losses. The optimization/valuation problem of standard swing contracts is not a trivial problem per se, as sub-period decisions typically impact the possibility of exercising the option in the future due to annual volume constraints. Thus, in the recent years swing options received vast treatment in the literature (see for instance [5, 19] and references therein for what concerns gas markets, and references in [3, 4, 6, 13] for swing options in more general markets).

The presence of make-up clauses further complicate things and introduces more complexity. Surprisingly, the quantitative literature appears scarce: in the authors’ opinion, this is due to the fact that a make-up clause is worth more in a market where price decoupling is high, and the need to study such markets arose only in the last years. At a qualitative level, for instance, the make-up clause is described in [21, 23]. An algorithm to solve the problem with the carry-forward clause is presented in [13], where the authors claim that the make-up can be evaluated similarly using the least square Monte Carlo approach, but the algorithm for the make-up is not presented. We furthermore believe that the make-up clause, having a two-times mechanism for the payment of postponed gas, cannot be priced simply by adapting an algorithm designed for the carry-forward clause. To the authors’ knowledge, an algorithm to properly price (and find out optimal policy) a swing contract with make-up clause and where both the market and strike price are stochastic variables have never been presented so far. The aim of this paper is exactly to fill this literature gap. In particular, we will describe, frame and solve the optimization issues related to the presence of make-up clauses in long term swing contracts. Finally, we use the algorithm in order to explore the value of the contract with respect to the peculiar constraints introduced by the make-up.

The paper is organized as follows. In Sections 2.1 and 2.2 we outline the generic structure of gas swing contracts and in Section 2.3 we describe a particular instance of make-up clause. Then in Section 2.4 we mathematically frame the problem and indicate an algorithm for its solution, describing analytically both its formal representation and
the various steps for the solution. In Section 2.5 we discuss the computational cost of our approach and obtain a quadratic cost with respect to the duration in years of the make-up clause. In Section 3 we extend our approach to another form of make-up clause as well as to some instances of carry-forward clause. In Section 4 we present a detailed example for a make-up clause of 3 years, and in Section 5 we use this contract to perform a sensitivity analysis in order to outline the key drivers for optimization and value protection given the current gas market scenario: in order to perform this analysis we calibrate two mean-reverting trinomial models to market data (in particular, TTF for the price $P$ and ENIGR07 for the index $I$) as explained in Appendix A. Concluding remarks in Section 6 end the paper.

2 The structure of swing contracts and make-up clause

2.1 Time structure and admissible strategies

Ordinary swing contract schemes are normally defined dividing each one of the $D$ yearly delivery periods $\{[T_j-1, T_j]\}_{j=1}^D$ into $N$ sub-periods $\{[t_{j,i-1}, t_{j,i}]\}_{i=1}^N$ obtaining the sequence $\{t_{j,i}\}$ such that

$$0 = T_0 = t_{1,0} < t_{1,2} < \ldots < t_{1,N} = T_1 = t_{2,0} < t_{2,1} < \ldots$$

$$\ldots \ < t_{j,i} < \ldots < t_{j,N} = T_j = t_{j+1,0} < \ldots < t_{D,N} = T_D$$

In particular, in every year $[T_{j-1}, T_j]$ we have the $N + 1$ points $(t_{j,i})_{i=0,...,N}$ such that $t_{j,0} = T_j$ and $t_{j,N} = T_{j+1}$.

We are also assuming that $N$ is also the number of exercise swing rights the holder has in every year, which can be exercised exactly at the points $t_{j,i}$, for $i = 0, \ldots, N - 1$ i.e. at the beginning of every sub-period. For example if the decisions are taken month by month, at the beginning of evey month, $N = 12$, if day by day $N = 365$.

Denote by $u_{j,i}$ the quantity of gas the holder decides to buy in the sub-period $[t_{j,i}, t_{j,i+1})$, $i = 0, \ldots, N - 1$, and by $z_{j,i}$ the cumulated gas quantity at time $t_{j,i}$. In particular we set $z_{j,0} = 0$ for all $j = 1, \ldots, D$ and

$$z_{j,i+1} = \sum_{k=0}^{i} u_{j,k} = z_{j,i} + u_{j,i} \quad \forall i \in \{0, \ldots, N - 1\}$$

(1)

Over each one of the $N$ sub-periods, minimum ($mDQ$) and maximum ($MDQ$) delivery quantities are established in the contract, which usually reflect physical effective transportation capacity limitations: thus, the quantities $u_{j,i}$ are constrained by

$$mDQ \leq u_{j,i} \leq MDQ \quad \forall i = 0, \ldots, (N - 1), \quad \forall j = 1, \ldots, D$$

(2)

For every contractual year, minimum and maximum quantities are also established, called respectively minimum annual quantity ($mAQ$) and annual contract quantity
The difference between the maximum gas that the holder could physically take and his contract right is thus given by

\[ \mathcal{M} := N \cdot MDQ - ACQ \]  

(3)

while the difference between the minimum gas that the holder must take by contract and the minimum which he could physically take is given by

\[ \mathcal{M} := mAQ - mDQ \cdot N \]  

(4)

Often we have non-trivial volume constraints, in the sense that

\[ \mathcal{M} > 0, \quad \mathcal{M} > 0 \]  

(5)

Thus, in the light of the discussion above, without any additional clauses and with non-trivial constraints we have

\[ N \cdot mDQ < mAQ \leq z_{j,N} \leq ACQ < N \cdot MDQ \]  

\[ \forall j = 1, \ldots, D \]

Penalty payments can be imposed if the volume constraints are exceeded in order to stimulate the buyer to respect the volumetric limits imposed (see for example [5]), but in this paper we do not take into account these penalties.

The difference between swing contracts with trivial and non-trivial volume constraints is extremely important in the pricing and hedging of the contract itself. In fact, with non-trivial volume constraints the holder must take into account, at time \( t_{j,i} \), not only the quantity \( u_{j,i} \) which would be optimal for that period, but also the effects of this quantity on the future decisions that he will be allowed to take after. This brings to model the so-called space of controls, i.e. the set where \( u_{j,i} \) is allowed to take values, in the following way. For a given year \( j = 1, \ldots, D \), assume that we have a final constraint \( z_{j,N} \in [\underline{z}, \overline{z}] \) for some \( 0 \leq \underline{z} < \overline{z} \). (\( [\underline{z}, \overline{z}] = [mAQ, ACQ] \) in the absence of make-up or other clauses). Then, for a given time \( t_{j,i} \), the space of controls \( A(t_{j,i}, z_{j,i}, [\underline{z}, \overline{z}]) \) will in general depend on time \( t_{j,i} \), cumulated quantity \( z_{j,i} \) and \( [\underline{z}, \overline{z}] \).

By the constraints (2) and construction of \( z_{j,i} \) at time \( t_{j,i} \), we can restrict our attention to the case when \( z_{j,i} \) satisfies the constraints

\[ mDQ \cdot i \leq z_{j,i} \leq MDQ \cdot i \]  

\[ \forall i = 0, \ldots, N \]

and

\[ N \cdot mDQ \leq \underline{z} \leq \overline{z} \leq N \cdot MDQ \]

The problem of determining the set \( A_{j,i} \) is non-trivial when Eq. (5) holds, which translates in

\[ N \cdot mDQ < \underline{z} \leq \overline{z} < N \cdot MDQ \]

(otherwise we can always reach the values in \( [N \cdot mDQ, N \cdot MDQ] \)). In this non trivial case, we are not allowed to take \( u_{j,i} = mDQ \) for all \( i = 0, \ldots, N - 1 \); in fact, there exists a time \( \tau_1 \) such that, if we have always took this minimum for \( t \leq \tau_1 \), then for \( t > \tau_1 \) we
have to switch to $u_{j,i} = \text{MDQ}$ in order to reach $\bar{z}$. This point $\tau_1$ is the common point
between the two lines $z = \text{MDQ}(t - t_{j,0})$ and $z = \text{MDQ}(t - t_{j,N}) + z$, $\forall t \in [t_{j,0}, t_{j,N}]$. A
simple calculation leads to

$$z_{j,i} \geq r_{\min}(t_{j,i}, \bar{z}) = \max \{\text{MDQ}(t_{j,i} - t_{j,0}), \text{MDQ}(t_{j,i} - t_{j,N}) + z\}$$

Similarly, we are not allowed to take always $u_{j,i} = \text{MDQ}$ either: in fact, there exists a
time $\tau_2$ such that, if we have always took this maximum for $t \leq \tau_2$, then for $t > \tau_2$ we
have to switch to $u_{j,i} = \text{MDQ}$ in order to reach, and not exceed, $\bar{z}$. The boundary for $z_{j,i}$
in this case is

$$z_{j,i} \leq r_{\max}(t_{j,i}, \bar{z}) = \min \{\text{MDQ}(t_{j,i} - t_{j,0}), \text{MDQ}(t_{j,i} - t_{j,N}) + \bar{z}\}$$

Figure 3 shows an example of the admissible area.

Figure 3: Typical admissible area for one year. Here $\underline{z} < \bar{z}$, leaving some optionality for
the total intake $z_{j,N}$. If $\underline{z} = \bar{z}$ (typical of years when some make-up gas is nominated or
called back), we have the constraint $z_{j,N} = \underline{z} = \bar{z}$ and the admissible region is like those
in Figure 4.

In conclusion, the correct form of the space of controls $\mathcal{A}(t_{j,i}, z, [\underline{z}; \bar{z}])$ at time $t_{j,i}$,
given the constraint $z_{j,N} \in [\underline{z}, \bar{z}]$ and the cumulated quantity $z_{j,i} = \underline{z}$, is given by

$$\mathcal{A}(t_{j,i}, z, [\underline{z}; \bar{z}]) := \{u_{j,i} \in [\text{MDQ}, \text{MDQ}] \mid z + u_{j,i} \in [r_{\min}(t_{j,i+1}, \underline{z}), \max(t_{j,i+1}, \bar{z})]\} \quad (6)$$
which appears implicitly in [3, Equation 7] and is also a discretized version of the one in [6].

2.2 The price of a standard swing contract

We now present a standard procedure to price a swing option without the presence of additional clauses (such as make-up).

Let $P_{j,i}$ and $I_{j,i}$ be respectively the prices of gas and index in year $j = 1, \ldots, D$, sub-period $[t_{j,i}, t_{j,i+1}]$, $i = 0, \ldots, N - 1$: the contract holder has to buy the gas at the price $I_{j,i}$ and can sell it at the price $P_{j,i}$: of course with this notation we have $(P_{j,N}, I_{j,N}) = (P_{j+1,0}, I_{j+1,0})$ for each year $j = 1, \ldots, D - 1$. Even if $I_{j,i} = (I_{j,i})_{j=1}^{D-1}$ is a time average on several subperiod and thus has relevant memory effects, as mentioned in the Introduction, in the following we make the simplifying assumption that $(P_{j,i}, I_{j,i})_{j=1}^{D-1, i=0} \ldots, N$ evolve as a two-dimensional Markov process under a pricing measure $P$, which is used in all the mathematical expectations that follow, while the numerical implementation that we use for the analysis of the next section will make use of the particular specification that we describe in Appendix A. We also assume that $E[e^{-rt_{j,i}} P_{j,i}] = F_{P_{j,i}} < +\infty$ and $E[e^{-rt_{j,i}} I_{j,i}] = F_{I_{j,i}} < +\infty$, where $F_{P_{j,i}}$, $F_{I_{j,i}}$ represent the forward prices of $P$ and of $I$, respectively, for the delivery time $t_{j,i}$.

The objective of contract’s holder is to maximize the discounted global margin of the contract (i.e. minimize the total loss), i.e., (s)he wants to calculate the value of

$$V(0, p_1, 0, \ell_1, 0) = \sup_{u \in A} E \left[ \sum_{j=1}^{D} \sum_{i=0}^{N-1} e^{-r t_{j,i}} u_{j,i} (P_{j,i} - I_{j,i}) \right]$$

where the set $A$ of admissible controls is defined by

$$A := \{ (u_{j,i})_{j,i} \text{ adapted to } (P_{j,i}, I_{j,i})_{j,i} \text{ and s.t. } u_{j,i} \in A(t_{j,i}, z_{j,i}, [mAQ, ACQ]) \}$$

in the absence of a make-up clause, and $r \geq 0$ is the risk-free annual interest rate. Equation (7) follows from the fact that $z_{j,0} = 0$ for all $j = 1, \ldots, D$, i.e. in the absence of a make-up clause the swing contract can be valued independently year by year.

It is a standard result (see e.g. [3],[5],[19]), and it will also follow as a particular case of our results in Section 2.4, that this maximisation problem can be solved by the use of the Dynamic Programming: for each year $j = 1, \ldots, D$, define the deterministic functions

$$V_j(N, p, \ell, z) := 0,$$

$$V_j(i, p, \ell, z) := \max_{u \in A(t_{j,i}, z, [mAQ, ACQ])} \left[ E^{P_{j,i}} \left[ e^{-r t_{j,i}} u (P - \ell) + V_j(i + 1, P_{j,i+1}, I_{j,i+1}, z + u) \right] \right] \forall i < N$$

(9)
where $\mathbb{E}^{P_{j,i}}$ indicates the expectation conditional to $P_{j,i} = p$ and $I_{j,i} = \iota$ (recall that, as these are Markov processes, these values are a sufficient statistics for the whole information up to subperiod $i$ of year $j$). Then the original problem in Equation (7) is brought back to calculating

$$V(0, p_{1,0}, \iota_{1,0}, 0) = \mathbb{E} \left[ \sum_{j=1}^{D} V_j(0, P_{j,0}, I_{j,0}, 0) \right]$$

### 2.3 Modeling the make-up clause

This subsection is devoted to the analytical representation of the make-up clause and its constraints. While long term contracts may have a length of 10-30 years, make-up clauses are typically written on a limited period of the contract life, often from 3 to 5 years. Given the fact that, as explained in Section 2.2, a contract without make-up clause can be evaluated as the sum of some yearly contract one independent from the other, we can split a contract with make-up written only on a subperiod of the whole contract life in two parts: the first part is a swing contract with a make-up clause with length equal to the original make-up clause, while the other part covers all the years when the make-up is not written. Thus, without loss of generality we can assume that the make-up clause is written on the whole contract’s length, $D$ years.

For each year $j = 1, \ldots, D$, call $M_j$ the make-up gas nominated and $U_j$ the make-up gas called back in year $j$. With this notation, we assume that the precise structure of the make-up clause follows these rules.

1. **For each year $j = 1, \ldots, D - 1$, the contract holder is allowed to take $z_{j,N} < mAQ$, provided $u_{j,i} \geq mDQ$ for all $i = 0, \ldots, N - 1$.**

Thus, the make-up gas nominated in year $j$ is

$$M_j := (mAQ - z_{j,N})^+ \quad \text{and must satisfy} \quad M_j \in [0, \mathcal{M}] \quad (10)$$

where $x^+ := \max(x, 0)$ and $\mathcal{M}$, defined in Equation (4), is also the maximum quantity of make-up gas that can be physically nominated in a given year.

2. **The make-up $M_j$ nominated in year $j$ can be called back in one or more subsequent years (the quantity $M_j$ can be split and called back in more than one year). This is possible only if the ACQ quantity has been reached in that year, and of course in that year we still have to satisfy $u_{j,i} \leq MDQ$ for all $i = 1, \ldots, N$.**

Thus, the make-up gas called back in year $j = 2, \ldots, D$ is

$$U_j := (z_{j,N} - ACQ)^+ \quad \text{and is such that} \quad U_j \in [0, \overline{\mathcal{M}}] \quad (11)$$

where $\overline{\mathcal{M}}$, defined in Equation (3), is also the maximum quantity of make-up gas that can physically called back in a given year.
3. It is not possible to call back make-up gas before having nominated it, and at year $D$ all the nominated make-up gas must have been called back.

Thus, if we define the cumulated gas debt at year $j$, i.e. the make-up gas not yet called back, as

$$M_j = \sum_{k=1}^{j} M_k - \sum_{k=2}^{j} U_k = \sum_{k=1}^{j} (M_k - U_k),$$

(12)

then $U_1 = M_D = 0$, $M_j \geq 0$ for all $j = 1, \ldots, D-1$ and $M_D = 0$. Moreover,

$$M_{j+1} = M_j + M_j - U_j = M_j + (\text{mAQ} - z_{j,N})^+ - (z_{j,N} - \text{ACQ})^+$$

Notice that conditions 2. and 3. imply, for example, that if at the beginning of the last contract year of the make-up clause we have some make-up gas not called back, i.e. $M_{D-1} > 0$, in year $d$ we necessarily have to reach the quantity $\text{ACQ} + M_{D-1}$.

**Remark 1** More in general, for all years $j = 1, \ldots, D$, the definition of $M_j$ implies that $M_j \leq j \cdot \bar{M}$ and $M_j \leq (D-j) \cdot \bar{M}$. By combining these two constraints, the maximum gas debt is possible at year

$$\bar{j} := \frac{D \bar{M}}{\bar{M} + \bar{M}}$$

(13)

if $\bar{j}$ is integer, and at one of the two nearest years if $\bar{j}$ is not integer. In particular, the gas debt $M_j$ can increase without constraints for $j < \bar{j}$ and must possibly be decreased for $j > \bar{j}$.

4. The price of the make-up quantity nominated in year $j$ and called back in year $k$, subperiod $i$, is defined as the weighted sum of two components respectively paid at two different times:

a) at time $t_{j,N}$ (i.e. at the end of year $j$ when $M_j$ becomes known) the buyer pays the make-up gas at the price $\alpha \bar{\Gamma}_j$ for some $\alpha \in (0,1)$ defined in the contract, where $\bar{\Gamma}_j$ is the average index price observed in year $j$;

b) at time of withdrawal $t_{k,i}$, the price paid is $(1-\alpha)I_{k,i}$.

The price of make-up gas, as defined above, is associated to the gas volume $u_{k,i}$ physically delivered at time $t_{k,i}$. This means that the part $\alpha \bar{\Gamma}_j$ in (a) of the price needs to be capitalized from time $T_j = t_{j,N}$ up to time $t_{k,i}$: thus, the price $I_{j,k,i}$ at time $t_{k,i}$ of the make-up gas nominated in year $j$ and called back in year $k$, subperiod $i$ is

$$I_{j,k,i} = \alpha \bar{\Gamma}_j e^{r(t_{k,i} - t_{j,N})} + (1-\alpha)I_{k,i}$$

(14)

By discounting at time $T_0 = 0$ the price of make-up gas called back at time $t_{k,i}$, we have

$$e^{-rt_{k,i}}I_{j,k,i} = \alpha \bar{\Gamma}_j e^{-rt_{j,N}} + (1-\alpha)I_{k,i}e^{-rt_{k,i}}$$
It follows that, in a year $j = 1, \ldots, D$ where the make-up clause is exercised to nominate or call back gas, the residual value of the swing contract at subperiod $i = 0, \ldots, N - 1$ for that year with the control policy $u_j := (u_{j,i})_{i=0, \ldots, N-1}$ is given by

$$J_j(i, p_j, \iota_j, z_j; u_j) := \mathbb{E}^{p_j, \iota_j}_{i,j} \left[ \sum_{k=i}^{N-1} e^{-r_j t_{j,k}} u_{j,k} \left( P_{j,k} - I_{j,k} \left( 1 - \alpha \mathbb{1}_{\left\{ z_{j,k} > \text{ACQ} \right\}} \right) \right) - e^{-r_j t_{j,N}} \alpha \Gamma_j M_j \right]$$

Here, the main apparent difference with respect to the case when no make-up is exercised is that we can end up a year with a non-null position in the make-up gas, i.e. with $M_j - U_j \neq 0$ (notice that this notation is not ambiguous as $M_j$ and $U_j$ cannot be both different from zero), where the quantity $M_j - U_j$ is by definition a deterministic function of $z_{j,N}$, thus of $u_j$. Notice that this generalizes the payoff to be maximised in Equation (7), which is reobtained for $z_{j,N} \in [m\text{AQ}, \text{ACQ}]$, i.e. $M_j = U_j = 0$, and setting $z_{j,0} = 0$.

As a result, the total value of the swing option with the make-up clause described above is given by

$$\sup_{u \in A} \mathbb{E} \left[ \sum_{j=1}^{D} J_j(0, p_j, \iota_j, 0; u_j) \right]$$

(16)

where $u = (u_{j,i})_{j=1, \ldots, D, i=0, \ldots, N-1}$ must now belong to the set

$$A := \left\{ (u_{j,i})_{j,i} \text{ adapted to } (P_{j,i}, I_{j,i})_{j,i} \mid u_{j,i} \in [\text{mDQ}, \text{MDQ}], \right. \left. U_1 = M_D = 0, M_j \geq 0 \forall j = 1, \ldots, D, M_D = 0 \right\}$$

and the functions $J_j$ are given by Equation (28). Here, the constraints on $U$ and $M$ induce constraints on $(u_{j,i})_{j,i}$, which will be treated in detail in the next subsection.

### 2.4 The price of swing contracts with make-up clauses

We just saw that the set of admissible strategies for $(u_{j,i})_{j,i}$ for a given year $j = 1, \ldots, D$ now depend on the gas debt $M_{j-1}$ arriving from the previous years. For this reason, this quantity has to be explicitly taken into account in the evaluation of the swing contract for that year. More in detail, now we define the value function as

$$V_j(i, p, \iota, z, M_{j-1}) := \sup_{u \in A} \mathbb{E}^{p, \iota}_{i,j} \left[ J_j(i, p, \iota, z; u_j) + \sum_{k=j+1}^{D} J_k(0, P_k, 0, 0; u_k) \right]$$

(17)
We now build a Dynamic Programming algorithm as in [5, 19]: for each year $j = 1, \ldots, D$, define the deterministic functions:

- if $j = D$, then $M_D = 0$ and $U_D = M_{D-1}$ (recall that $M_D \equiv 0$), so we let
  \[ z := mAQ_{1(M_{D-1}=0)} + (ACQ + M_{D-1})1_{\{M_{D-1}>0\}}; \quad \tau := ACQ + M_{D-1} \]  \tag{18} \]
  and define
  \[ V_j(N, p, \iota, z, M_{j-1}) := 0, \] \tag{19} \]
  \[ V_j(i, p, \iota, z, M_{j-1}) := \max_{u \in A(t_{j,i}, i, z, z)} E^{p, \iota}_{j,i} \left[ e^{-r t_{j,i} u (p - \iota (1 - \alpha 1_{\{z > ACQ\}}))} + V_j(i + 1, P_{j,i} + 1, I_{j,i} + 1, z + u, M_{j-1}) \right] \] \tag{20} \]

Notice that the functions $V_j$ depend on $M_{j-1}$ through $[z, \tau]$.

- for $j = 1, \ldots, D-1$ the key quantity is $M_{j-1}$ which is known at the beginning of the year. Assume that $M_{j-1} \leq (D - j + 1)M$. For the lower bound we have two cases.

  - If $M_{j-1}$ is admissible also for year $j$, i.e. if $M_{j-1} \leq (D - j)M$, then we can nominate some other make-up gas $M_j$ as long as
    \[ M_j = M_{j-1} + M_j \leq (D - j)M \Rightarrow M_j \leq (D - j)M - M_{j-1} \]
    Taking into account the mDQ constraints, the lower bound for $z_{j,N}$ is:
    \[ z := mAQ - \min \{(D - j)M - M_{j-1}, M\} \] \tag{21} \]
  
  - If $M_{j-1}$ is not admissible for year $j$, i.e. if $M_{j-1} > (D - j)M$, then we must call back some make-up gas in order to obtain a final cumulated quantity $M_j$ admissible for year $j$, i.e.
    \[ M_j = M_{j-1} - U_j \leq (D - j)M \Rightarrow U_j \geq M_{j-1} - (D - j)M \]
    So the lower bound for $z_{j,N}$ is now
    \[ z := ACQ + M_{j-1} - (D - j)M \] \tag{22} \]

For the upper bound $\tau$, we do not need to distinguish between the two previous cases and let
\[ \tau := ACQ + \min \{M_{j-1}, M\} \] \tag{23} \]

Finally, define
\[ V_j(N, p, \iota, z, M_{j-1}) := V_{j+1}(0, p, \iota, 0, M_{j-1} + (mAQ - z)^+ - (z - ACQ)^+) + e^{-r t_{j,N} \alpha \Gamma_j(mAQ - z)^+}, \] \tag{24} \]
and for $i = N - 1, \ldots, 0$ define $V_j(i, \cdot, \cdot, \cdot)$ exactly as Equation \[20\].
Theorem 2.1  
1. The deterministic functions $V_j(\cdot, \cdot, \cdot, \cdot, \cdot)$, defined by the dynamic programming equations (19), (20) and (24) are such that $V_j(0, P_{j,0}, I_{j,0}, 0, 0)$ coincides with the value of the swing option with the make-up clause in Equation (16).

2. There exists an optimal Markovian consumption $u^*_{j,i} = u(t_{j,i}, P_{j,i}, I_{j,i}, z_{j,i}, M_{j-1})$, where $u(\cdot, \cdot, \cdot, \cdot, \cdot)$ is given by the maximum argument in the dynamic programming equation (20).

3. If the quantities $K := M_{MDQ - mDQ}$ and $K := M_{MDQ - mDQ}$, (25) are integer, then there exists an optimal bang-bang Markovian consumption $u^*_{j,i}$, i.e. $u^*_{j,i} = mDQ$ or $u^*_{j,i} = MDQ$ for all $j = 1, \ldots, D$, $i = 0, \ldots, N - 1$. Moreover, $M_j$ turns out to be an integer multiple of $MDQ - mDQ$ for all $j = 1, \ldots, d$.

Proof. We proceed in analogy with [5] and [4].

1. As $0 \leq t_{j,i} \leq T$, $0 \leq z \leq N \cdot MDQ$, $M \leq D \cdot M$ and $E[P_{j,i}] = F_{j,i}^P < +\infty$, $E[I_{j,i}] = F_{j,i}^I < +\infty$, then the assumptions $(F^+, F^-)$ in [7, Proposition 8.5] are satisfied, so the argument follows.

2. The right-hand side of Equation (20) is continuous in $u$ and $A(t_{j,i}, z, [\bar{z}, \bar{z}])$ is a compact set contained in $[mDQ, MDQ]$, thus the maximum is attained for $u \in A(t_{j,i}, z, [\bar{z}, \bar{z}])$ again by applying [7, Proposition 8.5].

3. As in [4], it can be proved that the functions $V_j(i, \cdot, \cdot, \cdot, \cdot)$, $j = 1, \ldots, D$, $i = 0, \ldots, N - 1$ are continuous and concave on $z$ and piecewise affine on the intervals

$$[k \cdot mDQ + (i - k) \cdot MDQ, (k + 1) \cdot mDQ + (i - k - 1) \cdot MDQ], \quad k = 1, \ldots, i.$$ (26)

We now prove the claim by induction on $j = 1, \ldots, D$. If $j = 1$, then $M_0 = 0$ by definition. For a given year $j = 1, \ldots, D$, assume for now that $M_{j-1}$ is an integer multiple of $MDQ - mDQ$. Then this, together with the condition $K, \bar{K} \in \mathbb{N}$ ensures that $[r_{\min}(t_{j,i}, \bar{z}), r_{\max}(t_{j,i}, \bar{z})]$ is exactly the union of suitable intervals of the kind of Equation (26). Thus, if $z = k \cdot mDQ + (i - k) \cdot MDQ$ for some $k = 0, \ldots, i$, then the function to be maximised in Equation (20) is affine on $u$, thus its maximum point is $u^*_{j,i} = mDQ$ or $u^*_{j,i} = MDQ$. It can then be proved by induction that, since $z_{j,0} = 0$, the optimal $u^*_{j,i}$ is such that $z_{j,i} = k \cdot mDQ + (i - k) \cdot MDQ$ for some $k = 0, \ldots, i$; this also implies that $M_j$ will be also an integer multiple of $MDQ - mDQ$, and the conclusion follows.
Remark 2 Part 3. of the theorem above is essentially a consequence of the linear structure of the payoff function in Equations (28–16): the result is that in every year \( j \), subperiod \( i \), the optimal quantity \( u_{j,i} \) can be safely chosen to be either the maximum (MDQ) or the minimum (mDQ) admissible for that substep. This kind of control is called of \textit{bang-bang} type, and it was already found in [5] with smoother payoffs, and studied in deep detail in [4]. Qualitatively, this is due to the fact that, if the withdrawal is profitable in the subperiod, then the better choice is the maximum quantity we can take; conversely, if the withdrawal is not profitable, then the better choice is to take the minimum quantity we can.

2.5 Computational cost

As we have seen, the pricing problem for a swing option with make-up clause boils down to maximize the problem in Equation (17). Unfortunately, this maximization cannot be carried out by analytic means, as a closed form for \( V_j \) is not known even in the simplest case of a standard swing option without make-up clause. Thus, this maximization must be carried out via numerical methods.

The most efficient way to do this is to assume that the quantities \( K \) and \( \bar{K} \) in Equation (25) are integer, so that the results of Theorem 2.1 hold. This induces a quantization in the candidate optimal make-up gas debt \( (M_j)_{j=1,\ldots,D} \): in fact, since this process at optimality has values which are multiple integers of \( \text{MDQ} - \text{mDQ} \), we obtain that the resulting candidate optimal quantities for \( M_j, j = 1, \ldots, D \), are a finite number. More in detail, the sequence \( (M_j)_{j=1,\ldots,D} \) is bound to have a finite number of nonnegative values in each year \( j = 1, \ldots, D - 1 \), with \( -M \leq M_j - M_{j-1} \leq M \), i.e. the increments can have at most \( K + \bar{K} + 1 \) distinct values, corresponding respectively to the cases when \( M_j > 0, U_j > 0 \) and \( M_j = U_j = 0 \).

With this in mind, we can calculate the computational cost needed to price a \( D \)-year swing option with make-up clause, and we do this by the same backward recursion used in the Dynamic Programming algorithm used in Section 2.4. In the \( D \)-th year, we can start with \( M_{D-1} \) taking at most \( \bar{K} + 1 \) different values, each one of this leading to a different optimization problem: since also the values of \( (z_{j,i})_{j,i} \) are quantized via the bang-bang optimal process \( (u_{j,i})_{j,i} \), for each one of this optimization problem we have a total of \( O(N^2) \) states which can be assumed by \( (z_{j,i})_{j,i} \) at optimality. Having solved the \( \bar{K} + 1 \) problems for the last year, we can attach the value functions thus obtained to the terminal nodes of year \( D - 1 \): notice that also in this case, as now \( M_{D-2} \) can assume at most \( 2\bar{K} + 1 \) distinct values, we will have to model and solve at most \( 2\bar{K} + 1 \) distinct optimization problem, each one having as terminal condition the values of the \( \bar{K} + 1 \) problems of year \( D \). These numbers do not multiply, because once that we obtain the values for the \( \bar{K} + 1 \) problems of year \( D \) for each possible starting state \( (P_{D-1,N}, I_{D-1,N}) = (P_{D,0}, I_{D,0}) \), we can take these values as terminal values to use in the computation for year \( D - 1 \).

\footnote{Precisely \( \frac{N(N+1)}{2} \), which is the number of nodes of a complete recombining binomial tree.}
With this spirit, we are now ready for a result on the computational cost of the pricing of a swing option with make-up clause.

**Theorem 2.2** If the quantities $K$ and $\bar{K}$, defined in Equation (25), are integer, then the order of distinct subproblems to be solved is $O(N^2D^2)$.

**Proof.** First of all consider the 2-dimensional process $(j, M_j)_{j=1,\ldots,D}$: then the distinct states that this process can assume, at optimality, is in 1-1 correspondence with the integer solutions $(x, y)$ of the system

$$
\begin{cases}
  x \geq 0, \\
  x \leq \bar{K}(D - y), \\
  x \leq Ky,
\end{cases}
$$

where $\bar{j}$ denotes the integer part of $\bar{j}$. By noticing that for all $x > 0$ we have $D[x] \leq \lfloor Dx \rfloor$ and that

$$
\bar{j} = \frac{DK}{K + \bar{K}}
$$

where $\lfloor x \rfloor$ denotes the integer part of $x$. By noticing that for all $x > 0$ we have $D[x] \leq \lfloor Dx \rfloor$ and that

$$
\bar{j} = \frac{DK}{K + \bar{K}} = \bar{K}(D - \bar{j})
$$

then we have

$$
N \leq D + 1 + \frac{1}{2} \left[ \frac{DK \bar{K}}{K + \bar{K}} \right] (\bar{j} + 1 + [D - \bar{j} - 1]) \leq \frac{1}{2} \frac{K \bar{K}}{K + \bar{K}} D^2 + D + 1
$$

i.e. $N = O(D^2)$. By multiplying this for $O(N^2)$ (the number of subproblems for given year $j = 1, \ldots, D$ and state of make-up debt $M_j$, we obtain that the computational cost
is of order $O(N^2 D^2)$, as desired.

We show in Figure 4 an illustration of these numbers for $D = 2, 3, 4$, by making the simplifying assumption that $K = \overline{K} = K$.

![Figure 4](image)

Figure 4: In subfigure (a), we can only obtain $M_1 (= M_1 = U_2)$ among $K + 1$ distinct values, and the corresponding value function is then used in the final values of the optimization problem of year 1, so the total number of optimization problems to be solved is $K + 1$. In subfigure (b) we can obtain $M_2 (= U_3)$ among $K + 1$ distinct values and then $M_1 (= M_1)$ among $K + 1$ distinct values, so the number of optimization problems to be solved in sequence is $2K + 2$. In subfigure (c) we can obtain $M_3 (= U_4)$ among $K + 1$ different values, $M_2$ among $2K + 1$ different values, and finally $M_1 (= M_1)$ among $K + 1$ distinct values: so the number of optimization problems to be solved in sequence is now $4K + 3$.

3 Extensions of the model

This section is devoted to show some extensions of the algorithm proposed so far in two directions. The first one is towards another form of make-up clause, while the second one is directed to a possible application to carry-forward clauses.

3.1 Another form of make-up clause

While different contracts may have several slightly different definition of the make-up clause, up to authors’ knowledge the most negotiated variant to the make-up clause
presented in Section 2.3 is obtained as in \[21\] by modifying Rule 2 as follows: the make-up gas must be called back as soon as \(mAQ\) has been reached instead of after having exceeded \(ACQ\). In this case every time that at the beginning of the year the cumulated make-up quantity \(M_{j-1}\) is positive then all the quantity exceeding \(mAQ\) is considered to be make-up gas called back.

The maximum make-up quantity we are allowed to call back every year is now

\[
\mathcal{M} = ACQ - mAQ
\]

and again we can call it back only if \(M_{j-1} > 0\).

The algorithm for this specification of the make-up clause is similar to the one presented in Section (2.4), provided we redefine some of the quantities. Precisely, redefine

\[
U_j := 1_{\{M_{j-1} > 0\}} (z_{j,N} - mAQ)
\]

which must be such that \(U_j \in [0, \mathcal{M}]\), where the maximum make-up quantity which can be called back in a given year is now defined as \(\mathcal{M} := ACQ - mAQ\). This of course modifies the payoff in Equation (28), which now becomes

\[
J_j(i, p, \iota, z; u_j) := E_{p, \iota}^j \left[ e^{-rt_j} u_j(P_j - I_j - (1 - \alpha) 1_{\{z > mAQ, M_{j-1} > 0\}}) \right] - e^{-rt_j} \alpha \Gamma_j M_j
\]

The Dynamic Programming algorithm has to be modified as follows. Substitute Equation (18) with

\[
z := mAQ + M_{D-1}, \quad \bar{z} := ACQ 1_{\{M_{D-1} = 0\}} + (mAQ + M_{D-1}) 1_{\{M_{D-1} > 0\}},
\]

Equation (20) with

\[
V_j(i, p, \iota, z, M_{j-1}) := \max_{u \in A_{i, \iota, z, [\bar{z}]}(\bar{z})} E_{p, \iota}^j \left[ e^{-rt_j} u \left( p - \iota \left(1 - \alpha 1_{\{z > mAQ, M_{D-1} > 0\}}\right) \right) + V_j(i + 1, P_j, I_j, z + u, M_{j-1}) \right],
\]

Equation (22) with \(\bar{z} := mAQ + M_{j-1} - (D - j)\mathcal{M}\), Equation (23) with \(z := ACQ\), and finally Equation (24) with

\[
V_j(N, p, \iota, z, M_{j-1}) := V_{j+1}(0, p, \iota, 0, M_{j-1} + (mAQ - z)^+ - (z - mAQ)^+ 1_{\{M_{j-1} > 0\}}) + e^{-rt_j} \alpha \Gamma_j (mAQ - z)^+.
\]

With these substitutions, Theorem 2.1 still applies to this case.
3.2 Carry forward clauses

The approach used in this paper can also price another clause related to swing contracts, namely some instances of the carry-forward clause as described in [13]. In general, the carry-forward (CF) right gives the holder of the option the possibility to reduce mAQ (up to a contractual amount ACF called annual carry-forward) in one year if in at least one of the previous \( d \) years the total volume taken was above mAQ, while the maximum quantity which can be taken every year remains ACQ.

With a slight modification of the algorithm presented in this paper we are able to price two particular instances of the carry-forward clause presented in [13], namely when the CF rights must be used in the following year (corresponding to \( d = 1 \), and when CF rights do not have a deadline, this clause being called unlimited duration carry-forward (UDCF): in this latter case there is a maximum on the CF rights one can obtain in a given year, given by \( \nu \cdot ACF \), where \( \nu > 1 \) is a contractual constant.

We now present how to price a CF clause with \( d = 1 \). First of all, define the gas credit \( U_j \) at year \( j = 1, \ldots, D \) as

\[
U_j := (z_j, 12 - mAQ)^+ 
\]

and \( U_0 := 0 \), and the minimum and maximum cumulated quantities for year \( j = 1, \ldots, D \) as

\[
x := mAQ - \min(U_{j-1}, ACF), \quad \tau := ACQ
\]

Then the Dynamic Programming algorithm can be built as follows: for the last year \( D \) define

\[
V_D(N, p, t, z, U_{D-1}) := 0,
\]

while for \( j = 1, \ldots, D - 1 \) define

\[
V_j(N, p, t, z, U_{j-1}) := V_{j+1}(0, p, t, 0, (z - mAQ)^+),
\]

and for \( i = N - 1, \ldots, 0 \) define \( V_j(i, t, z, U_{j-1}) \) exactly as in Equation (30). Then \( V_l(0, p, t, 0, 0) \) gives exactly the value of the CF contract with \( d = 1 \).

We now present how to price a UDCF contract. First of all, redefine the cumulated gas credit \( U_j \) at year \( j = 1, \ldots, D \) as

\[
U_j := U_{j-1} + \min(\nu \cdot ACF, z_j, 12 - mAQ)
\]

with \( U_0 := 0 \): with this new definition, the gas credit \( U_j \) at the end of year \( j \) increases only if we exceed \( mAQ \) and decreases every time we do not reach \( mAQ \). Redefine also
the minimum and maximum cumulated quantities for year \( j = 1, \ldots, D \) as in Equation (28). Then the Dynamic Programming algorithm can be built as follows: for the last year \( D \) define the value function \( V_D \) exactly as in Equations (29–30), while for \( j = 1, \ldots, D - 1 \) define

\[
V_j(N, p, \iota, z, U_{j-1}) := V_{j+1}(0, p, \iota, 0, U_{j-1} + \min(\nu \cdot ACF, z_j, 12 - mAQ)),
\]

and for \( i = N - 1, \ldots, 0 \) define \( V_j(i, \cdot, \cdot, \cdot, \cdot) \) exactly as in Equation (30). Then \( V_1(0, p, \iota, 0, 0) \) gives now exactly the value of the UDCF contract.

**Remark 3** In principle, it would be possible to price also CF clauses with \( d > 1 \) by introducing other state variables, but this goes beyond the scope of this paper, which is focused on make-up clauses. Also, here the computational burden could be reduced by modelling the difference process \( P - I = (P_{j,i} - I_{j,i})_{j,i} \) as a single state variable as in [13]. Unfortunately, this is not possible with the make-up clause because of the delayed payment structure of the make-up gas.

### 3.3 Another possible approach

The Dynamic Programming approach algorithm is not the only algorithm one can pursue to solve this problem. Another approach could be the Least Square Monte Carlo (LSMC) method as presented in [13].

This method is based on the intuition that conditional expectations in the pricing algorithm can be replaced by the orthogonal projection on some space generated by a finite set of functions, obtained using Monte Carlo simulations and least-squares regressions to estimate numerically said orthogonal projection. While the LSMC algorithm is very flexible, on the other hand that approach may be influenced by a lot of user’s choices which may influence the pricing procedure, for instance the type and the number of basis functions and the number of Monte Carlo simulations used. These choices can be critical, as [22] shows that while for some type of derivatives (such as the American put) the LSMC approach is very robust, for more complex derivatives the number and the type of basis functions can slightly affect option prices. We are not stating that one algorithm is absolutely better than the other. In fact, for some type of very complex control problems LSMC algorithm represents probably the best possible solution one can pursue. Conversely, for other problems such as the swing contract with monthly granularity presented so far, where also the lattice approach works well, the decision between the two algorithm becomes a matter of what one decide to simplify.

### 4 Three years example

In the following we describe and analyze the case of a three years contract, i.e. the case \( D = 3 \) and \( N = 12 \). Although, as seen in Subsection 2.5, the complexity of the different kinds of control problems to be solved grows quadratically with \( D \), we have decided to present as a first example a three years contract due to fact that the distinct
qualitative combinations of years where we can nominate and/or call back make-up gas, as $D$ grows, lead to more and more intricate combinatorial considerations, which would deviate the attention from the modelization. Thus, while a 2-year contract is not a very interesting example from a modellistic point of view, as at the end of the first year the make-up quantity is known and is exactly the quantity called back in the second year, on the other hand we think that $D = 3$ gives the right compromise between succeeding to follow exactly what goes on in the different years and significance of the combinatorial problem.

We also have to remark that typical make-up clauses are not alive during all the life of the contract, but are typically written on small sub-period spanning from 3 to 5 years. At a first glance, this choice may seem strange considering the fact that swing contracts usually have longer maturity, from 10 to 30 years. The main reason why make-up clauses have significantly shorter duration is that no seller takes the risk of giving the opportunity to move huge quantities of gas between decades. In addition, European gas market illiquidity does not allow to obtain realistic forecast of forward prices. As an example, the longest traded maturity on a very liquid market as the TTF is the 3-year ahead forward, so we have no particular indication from the market for the long term gas term structure, which is thus pretty flat. In such situation the decoupling is less marked and the make-up clause loses a bit its importance. However, if need arises for the buyer, a new make-up clause can be renegotiated in future years, so the problem of valuing make-up clauses can be split into separate problems.

Let us now concentrate on our 3-year example. Once the make-up quantity of the first year is known, in the second year there are many opportunities: one can call back some (or all) the make-up of the first year or can nominate, if possible, some other make-up that will be called back in the third year.

4.1 Trees quantity

First Year. By Remark 1, we must end the year with

$$0 \leq M_1 = M_1 \leq \min(M, 2M)$$

In fact, in the first year the maximum make-up quantity we can nominate has to be less than or equal to the maximum quantity we can call back in the following two years, that is $2M$, as well as to the maximum quantity we can nominate in a single year. In terms of $z$ and $\tau$, this means that $z = mA - \min(M, 2M)$, while $\tau = ACQ$, because we can not call back any previous year make-up, so we have

$$z_{1,12} \in \left[mA - \min(M, 2M), ACQ\right]$$

Notice that this is in agreement with Equations (21) and (23). Figure 5 shows the possible actions we can perform in the first year.

Second Year. Notice that in this year $M_1 = M_1$: this value strongly influences the possible actions we can take in the second year:
If \((0 \leq M_1 \leq \overline{M})\) we can do one of the following:

i. nominate some other make-up gas in the second year, in such a way that we are able to call back all the make-up in the third year, that is

\[
M_1 + M_2 \leq \overline{M} \Rightarrow M_2 \leq \overline{M} - M_1
\]

ii. call back some make-up gas nominated in the first year: the maximum quantity we can call back is, obviously, \(M_1\);

iii. take a quantity of gas between \(mAQ\) and \(ACQ\), not nominating or calling back any make-up gas.

Summarizing, the constraints for \(z_{2,12}\) in this case are

\[
z_{2,12} \in [mAQ - (\overline{M} - M_1), ACQ + M_1]
\]

which are again in agreement with Equations (21) and (23). Figure 6(b) shows this case.

Figure 5: Tree quantities for the first year. In the final states where \(z_{1,12} < mAQ\), some make-up is nominated and has to be called back in the subsequent 2 years.
• if \( \overline{M} < M_1 \) (\( \leq 2\overline{M} \)) we must instead call back some make-up gas \( U_2 \), otherwise we are not able to arrive in \( T_3 \) having called back the whole quantity \( M_1 \). In this case the minimum \( U_2 \) we have to call back must be such that the final make-up cumulated quantity can be called back in the third year, i.e. \( M_2 \leq \overline{M} \), which leads to:

\[
M_2 = M_1 - U_2 \leq \overline{M} \Rightarrow U_2 \geq M_1 - \overline{M}
\]

Thus, the following constraints for \( z_{2,12} \) hold:

\[
z_{2,12} \in [ACQ + M_1 - \overline{M}, ACQ + M_1]
\]

which now are in agreement with Equations (22) and (23). Figure 6(a) shows this case.

---

**Third year.** The key quantity now is the residual make-up we have to call back, if any. This quantity, that is exactly \( M_2 \) as defined in Equation (12), is the residual make-up quantity we must call back in the third year, being this the last year contract (remember we have to call back all the nominated make-up, as seen in Subsection 2.3).

So there are two cases for this year:

- if \( M_2 > 0 \) then we have to call back the whole make-up quantity accumulated in the first two years and we have no choice for \( z_{3,12} \):

\[
z_{3,12} = ACQ + M_2
\]
This case is shown in Figure 7(a).

- if $M_2 = 0$ then, as we cannot nominate any make-up, the constraints for $z_{3,12}$ are given by:
  
  $$z_{3,12} \in [mAQ, ACQ]$$

  This case is shown in Figure 7(b).
  
Both the cases agree with Equation (18).

Figure 7: Possible tree quantities for the third year: here the kind of tree totally depends on the cumulated make-up residual quantity $M_2$ from the previous years. If $M_2 > 0$, we are forced to put $U_3 = M_2$ and consume $z_{3,12} = ACQ + U_3 > ACQ$, ending up with a tree as subfigure (a). If $M_2 = 0$, we are forced to satisfy the constraints and consume $z_{3,12} \in [mAQ, ACQ]$, ending up with the tree in subfigure (b).

5 Sensitivity analysis of a three years contract

A swing contract is a derivative product whose value depends on two main classes of factor, namely market and volumetric. As previously explained in this paper, this kind of derivative shows an optionality value linked to the market price dynamics of the underlying commodity (exercise or not) and an optionality value linked to the volumetric structure of the product itself (how much to allocate with the make-up clause among the years and how much to withdraw in each subperiod). After having explained how to price a swing product on gas and how to determine the optimal exercise policy, it is now interesting to use the algorithm in order to explore and map the value of the contract with respect to some peculiar parameters of the contract and to market factors.

More in detail, we specify a trinomial dynamics for both the price $P$ and the index $I$ which approximates a geometric mean-reverting Ornstein-Uhlenbeck process as described in Appendix A, and calibrate these models following [8], using historical data.
on TTF prices for the gas price $P$ and the ENIGR07 formula\[^4\] for the index price $I$. For ease of implementation, the average index price $\Gamma_j$ of year $j$ which appears in Equation (24) is substituted with the average of forward prices for that year. When not variable, the parameters used in this section are the ones in Table 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{ACQ}$</td>
<td>$7.00 \cdot 10^6$</td>
<td>$\sigma^P$</td>
<td>$0.6$</td>
</tr>
<tr>
<td>$m\text{AQ}$</td>
<td>$6.00 \cdot 10^6$</td>
<td>$a^P$</td>
<td>$2.95$</td>
</tr>
<tr>
<td>$\text{MDQ}$</td>
<td>$8.75 \cdot 10^5$</td>
<td>$\sigma^I$</td>
<td>$0.1$</td>
</tr>
<tr>
<td>$m\text{DQ}$</td>
<td>$3.75 \cdot 10^5$</td>
<td>$a^I$</td>
<td>$19.04$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$0.75$</td>
<td>$S$</td>
<td>$0$</td>
</tr>
<tr>
<td>$r$</td>
<td>$0.05$</td>
<td>$\rho$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 4: Values of the parameters used for the analysis (when not variable).

We here present three analyses: the first one with respect to the volatility level $\sigma^P$ of gas price, to the MDQ contract parameter, and to the level of market price decoupling. The second one is done with respect to the level of decoupling of the price term structure and to interest rates level. Finally, the third one is done with respect to correlation between $P$ and $I$ and level of decoupling.

The choice of these analyses have been done considering the aim of what we are pursuing, that is to analyse the flexibility given by the make-up clause in a decoupled market scenario. In view of this, we decided to change the parameters we believe to be more impactive on the value of the make-up clause. The volatility $\sigma^P$ is representative of market uncertainty: in fact, $\sigma^P$ is often much greater than $\sigma^I$, as the index $I$ is calculated as a time average of a basket; as mentioned in the Introduction, this averaging is used to reduce the volatility of the index and leads also to a pretty stable value for $\sigma^I$, so changes in $\sigma^P$ are likely to influence the price more than ones in $\sigma^I$. The choice of MDQ is explained by the fact that this quantity is strictly linked with the maximum make-up $M$ the owner of the contract can call back in every year. In fact, the bigger MDQ is, the bigger $M$ becomes, and higher the possibility of the owner becomes to posticipate the calling back of the nominated make-up gas. This flexibility should increase the contract value, in particular when price decoupling is strong. We have decided not to move the minimum quantities. On one hand we set the minimum annual quantity and the minimum period quantity in such a way that the possible make-up one can nominate every year is very high ($1.5 \cdot 10^6$), so the stronger constraints are on $\overline{M}$. On the other hand, we imposed the values of $\overline{K}, \overline{K}$ to be integer and we used values for MDQ in Table 5. The underlying idea is that any possible increase in the callable make-up quantity $\overline{M} = m\text{AQ} - N \cdot m\text{DQ}$ is worthless if the upper bound of gas withdrawal per year $\overline{M}$ is not enough to call back the nominated make-up quantity. Thus, we map the contract value for MDQ in the range between $\overline{ACQ} = 7.10^6 / 12 \approx 5.83 \cdot 10^5$, which reduces

\[^4\]The ENIGR07 (ENI Gas Release 2007) index is a 9-months time average of a basket of three oil-related indexes, computed as in [1] Equation (1) or in [13] Equation (1).
to the case of a standard contract without make-up clause\footnote{in fact, if if \( \text{MDQ} = \frac{\text{ACQ}}{N} \), then, being not possible to call back any make-up gas before having reached \( \text{ACQ} \), we are never able to call back any make-up gas, thus it is also impossible to nominate some.} and a value big enough to ensure the withdrawal in the third year of the possible make-up gas nominated in the first and second year, i.e. bigger than \( \frac{\text{ACQ} + 2(\text{mAQ} - N \cdot mDQ)}{N} \simeq 8.3 \cdot 10^3 \) and such that \( K, \bar{K} \) are integers.

<table>
<thead>
<tr>
<th>MDQ</th>
<th>M</th>
<th>M</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.83 \cdot 10^5</td>
<td>0</td>
<td>1.5 \cdot 10^6</td>
<td>No make up</td>
</tr>
<tr>
<td>6.25 \cdot 10^5</td>
<td>5 \cdot 10^5</td>
<td>1.5 \cdot 10^6</td>
<td>Low flexibility</td>
</tr>
<tr>
<td>8.75 \cdot 10^5</td>
<td>3.5 \cdot 10^6</td>
<td>1.5 \cdot 10^6</td>
<td>High flexibility</td>
</tr>
</tbody>
</table>

Table 5: Values of MDQ used in the analysis. All the other parameters, when not variable, are set as in Table 4.

The choice of changing MDQ and not other parameters is also a consequence of the practice: we think that the minimum annual quantity and the minimum period quantity are less negotiated than the maximum ones: the seller of the contract will never be willing to sell too much flexibility at the expense of its profits (he want to sell the physical gas), and the buyer will not pay too much for some flexibility (he needs the physical gas).

The second and third analyses mainly focus on market factors. As already stated, the make-up clause becomes profitable for the buyer of the contract only if the spread between market and index price \( P_t - I_t \) is expected to be lower in the future than in the present. On the other hand, the make-up gas is paid in two different times and its price is affected by the interest rate, as seen in Eq. (14). Consequently, the benefits of the decoupling could be affected by high levels of interest rates, which potentially may vanish the power of make-up clause. This is the focus of the second analysis. Also the correlation could potentially affect the benefits given by the decoupling: in principle, the decoupling should be enforced by negative correlation and weakened by positive one. This is the subject of the third analysis.

**First Analysis.** The first analysis studies how the contract value depends on the volatility level \( \sigma^P \), on the MDQ contract parameter and on the level of decoupling. The latter is obtained by varying the initial forward prices used to calibrate the tree prices (see Appendix A), subtracting a level \( S \) from the forward prices \( F^P \) for the first year and adding the same quantity to the forward prices for the third year, as shown in Subfigure 10(b). Then we let \( S \) be a parameter and see how the swing price depends on it.

We expect the swing contract value to be increasing in \( \sigma^P \), with a higher dependence when there is no flexibility given either by the absence of a make-up clause or by small values of MDQ. Figure 8 shows exactly these qualitative intuitions. The contract value is increasing with respect to \( \sigma^P \) also for high values of MDQ, but the range in \( y \) axes in
the figure is so large that we may not appreciate the monotonicity of the curves. This also evidences the fact that the rights given by make-up reduce the risk given by market uncertainty.

![Sensitivity with respect to $\sigma P$ for three values of MDQ.](image)

Figure 8: Sensitivity with respect to $\sigma P$ for three values of MDQ.

The dependence between contract value and decoupling parameter $S$ is presented in Figure 9: make-up rights are useful when market decoupling is high. In these situations, we can nominate make-up gas at the beginning of the contract life and call it back in the future, when a positive market scenario shows up.

**Second Analysis.** The second analysis is performed by mapping the swing value with respect to the decoupling parameter $S$ and the interest rate $r$ and reporting the corresponding prices in Figure 11.

The spirit of this analysis is that the make-up clause is exercised when a negative market scenario (typically, contractual price $I$ higher than spot gas price $P$) is expected to change or disappear in the following years through a change in the slope of the index and the gas price forward term structure. On the other hand, as we saw in Subsection 2.3, the make-up gas nominated is paid partly immediately, and partly when the gas is withdrawn; this temporal mismatch implies a cash flow effects whose impact obviously depends also on the interest rate level: for higher interest rate levels, the benefit of the make-up clause is absorbed by the capitalization of the cost sustained from the end of make-up nomination’s year up to the withdrawal period. Conversely, in a standard
contract without make-up clause, a higher interest rate in a market scenario with a low level of decoupling may lead to a higher contract value: in fact, if the decoupling is low, the present value of the contract in the long term, where the swing option is at or out of the money, is lower than the value in the short term, where the option is in the money. Figure 11 shows how any positive change in $S$ is negatively compensated by an increase in the interest rates level.

**Third Analysis.** The third analysis maps the contract value with respect to the correlation $\rho$ between the two prices $P$ and $I$, and the level of decoupling $S$. In Figure 12(a) we see that decoupling knocks out correlation: in fact, the swing price’s dependence on $S$ is much greater than that on $\rho$, enforcing once again a strong dependence of the swing price on decoupling levels. Only a deeper analysis, performed for fixed values of decoupling, allows a better understanding of the impact of correlation: negative values of $\rho$ leads to higher values of the contract. This is not a surprise: when $\rho$ is negative the decoupling between prices is expected to be stronger (if $P$ rises up then $I$ falls down thanks to $\rho < 0$) and this increases the value of the contract. However, the changes due to correlation are still smaller than the changes due to decoupling, even for small values of $S$. 

Figure 9: Sensitivity with respect to the decoupling $S$ and three values of MDQ, from no make-up rights to very large flexibility. As expected, decoupling enforces make-up value.
Figure 10: Scenarios for the term structure of gas and index prices for two levels of decoupling. In subfigure (a) make-up rights are typically not exercised, and prices are not decoupled, while in subfigure (b) typically make-up gas is declared in the first year and called back in the third year thanks to the decoupling.

6 Conclusions

The oil-to-gas price decoupling of the latest years, especially since the 2008 financial crisis onwards, made the make-up clause a very important feature embedded in most of long-term gas swing deals. In this paper we describe, frame and solve the optimization issue related to the presence of a make-up clause in a swing option. As for a standard swing contract, we show that it is possible to reduce the pricing of a swing option with make-up clause to a stochastic control problem, which can be solved using in a suitable way the Dynamic Programming algorithm. The key idea is to introduce the make-up gas debt as new state variable and incorporating it in the annual constraints on the state space. The dynamic programming is used both on every sub-period of the contract and year by year, by taking into account the gas debt at the beginning of every year. It turns out that, under some not very restrictive assumptions, the optimal withdrawal in all the single sub-periods is of bang-bang type, i.e. it is always optimal to choose in every sub-period between the minimum (mDQ) or the maximum (MDQ) possible withdrawal quantity. This induces a quantization on the number of distinct sequences of the different optimization problems we have to solve in every year, and this number is shown to be dependent on the range MDQ – mDQ and on the annual upper and lower bounds (ACQ and mAQ, respectively). We prove that the total number of optimization problems to solve is quadratic with respect to the product of the duration $D$ of the contract in years and the number $N$ of considered sub-periods. After having described the full algorithm for a generic number of years $D$, we extend this algorithm to another form of make-up clause, as well as to another clause possibly present in swing contracts,
Figure 11: Sensitivities with respect to \( r \) and level of decoupling \( S \) in the forward prices of \( P \). The first three cases on the top are with make-up, the last three in the bottom without.

The algorithm and its extensions are followed by a detailed description of a 3-years contract, which shows how the algorithm works in every year and how the problem is potentially complex even for small values of \( D \). The algorithm of Section 2 is implemented on this 3-years contract, by choosing as the dynamics of \( P \) and \( I \) a suitable trinomial model with mean-reverting properties, which is calibrated to market data (in particular, TTF for the price \( P \) and ENIGR07 for the index \( I \)) as explained in Appendix A. This implementation is then used to perform a sensitivity analysis of the price with respect to \( \Delta \) and some other market parameters, namely the volatility of the spot price \( \sigma_P \), the correlation between spot price and market index \( \rho \), the interest rates level \( r \), and the possible decoupling of gas and index prices induced by the term structure of forward prices, introduced as a perturbation modulated by a parameter \( S \) (see Figure 10).

The first conclusion is that the market uncertainty given by volatility can be decreased using make-up rights: high levels of make-up rights leads to a less marked dependence in contract value in contrast with the case of a standard contract, where the dependence is more pronounced.

The main conclusion is however that the decoupling induced by forward prices is crucial in assessing whether the make-up clause is a significant component in the price
Contract Value
Decoupling $\rho$ and level of decoupling. In Figure (a) the shift $S$ vanquishes the effect of the correlation: in fact by varying $\rho$ we obtain almost indistinguishable curves, both with or without make-up. In order to see the differences between curves, in Figure (b) the shift $S$ is fixed and here we can see how correlation affects contract value with make-up: negative values of $\rho$ lead to higher contract values (negative $\rho$ supports decoupling), but the stronger influence of the decoupling $S$ is always evident.

of the swing option. Figure 9 is clear: the higher is the value of the make-up rights, the higher is the dependence of the price on the decoupling. The slope of the contract value changes completely with high make-up rights, turning the decoupling into a favourable market behaviour. Large values of decoupling $S$ also increase the dependence of the swing price on interest rates: in Figure 12 we show how the make-up is sensible to high values of the risk free rate $r$. The benefits of the decoupling may be in contrast with high rates and make-up, but also in this cases a contract with make-up is always worth more than a contract without. Finally, we investigate the dependence of contract value with respect to correlation $\rho$. It turns out that contract value remains more or less unchanged when correlation changes, so this dependence is much less significant than the dependence on decoupling.

In conclusion, make-up clauses are a powerful tool to manage primary the new market scenario induced by decoupling and also the uncertainty given by prices. We expect that such type of contracts will be traded more frequently in the future, and here we presented a fast algorithm to price them.

Future work should mainly aim to properly take into account the discrete nature of the index price $I$, as opposed to the continuous evolution of gas price $P$, as well as the implicit non-Markovianity of $I$ (being it a time average based on several months, its evolution has relevant memory effects).
Acknowledgements.

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A Tree Prices

We assume that the log-prices \( X_{j,i} := \log P_{j,i} \) and \( Y_{j,i} := \log I_{j,i} \) follow the discretized version of the mean-reverting dynamics

\[
\begin{align*}
    dX_t &= (\theta^P - a^P X_t) \, dt + \sigma^P \, dW^P_t \\
    dY_t &= (\theta^I - a^I Y_t) \, dt + \sigma^I \, dW^I_t
\end{align*}
\]

where \( W^P_t \) and \( W^I_t \) are two Brownian motions with mutual correlation \( \rho \): these processes are particular cases of the model in [24] and are rather standard models for energy prices (see for example [14, Chapter 23.3].

In the discretized version, both \( X_{j,i} \) and \( Y_{j,i} \) change at the beginning of every sub-period (i.e. at the beginning of every month). This is exactly what happens for the index \( I_t \), and it is an acceptable simplification for the gas price \( P_t \). In particular, we discretize the prices \( (P_{j,i})_{j,i} \) and \( (I_{j,i})_{j,i} \) by building two trinomial trees with the procedure explained in [9, 14] and here summarized.

The first step is to build trinomial trees for \( X \) and \( Y \) by discretizing the dynamics of processes

\[
\begin{align*}
    dX^*_t &= -a^X_t \, dt + \sigma^X \, dW^X_t, \quad X^*_0 = 0 \tag{33}
\end{align*}
\]

with \( (a, \sigma) = (a^P, \sigma^P) \), or \( (a, \sigma) = (a^I, \sigma^I) \) in the analogous specification for \( Y^* \). The trees for these processes are symmetric around 0 and their nodes are evenly spaced in time and value at intervals of predetermined length \( \Delta t \) and \( \Delta X^* = \sigma\sqrt{3}\Delta t \).

As usual, we denote by \((i,j)\) the node \( x_{i,j} \) in the tree for which \( x_{i,j} = X^*_t \) with \( t = i\Delta t \) and \( X^*_{i\Delta t} = j\Delta X^* \)[6] Hull and White proved [15, 17] that the probabilities to switch from node \((i,j)\) to node \((i+1,k)\) are always nonnegative if \(-\bar{j} \leq j \leq \bar{j}\), where \( \bar{j} \) is the smallest integer greater than \( 0.184/(a\Delta t) \). This means that at every time step \( i = 0, \ldots, N \) we have a finite number of nodes \((i,j)\) placed at points \( j\Delta X^* \) for every integer \( j \in \{-j^*, \ldots, 0, \ldots, j^*\} \), with \( j^* := \min \{ \bar{j}, 2i - 1 \} \). Thus, the total width of the tree depends on \( a, \sigma \) and \( \Delta t \).

The second step is to put together the two trinomial trees in a 2-dimensional tree for \((X^*, Y^*)\): this is done at each node in such a way to preserve the marginal distributions.

\[\text{\footnote{Notice that in this Appendix the notation \((i,j)\) is not referred to the notation “year } j \text{, month } i \text{” used until now in the paper. Here we not distinguish between year and months, having a unique time index } i \text{ that varies between 0 and } N \cdot D. \text{ However, for sake of notation, in this appendix we suppose that } i = 0, \ldots, N, \text{ being } N \text{ the appropriate number.}}\]
of $X^*$ and $Y^*$ and the covariance structure induced by the correlated Brownian motions $W^p$ and $W^I$, as in [16] (see also [9] Appendix F).

The third step is aimed to calibrate the previous symmetric tree to the term structure $F_i$ one has, $F_i$ standing for the value of the forward with maturity $i \Delta t$: this step is used to incorporate into the tree mean reversion to levels different from zero, and in particular can be used here to introduce seasonality effects. This is obtained by adding a quantity $\alpha_i$ to the value $x_{i,j}$ of all nodes $(i,j)$. For every step $i$ we have a value for $\alpha_i$ such that:

$$
\sum_j Q_{i,j} e^{\alpha_i + x_{i,j}} = F_i
$$

that leads to

$$
\alpha_i = \log(F_i) - \log \left( \sum_j Q_{i,j} e^{x_{i,j}} \right)
$$

having denoting with $Q_{i,j}$ the probability to reach the node $(i,j)$ starting from the node $(0,0)$. Once we have the values for $\alpha_i$ we obtain the final tree which has, at step $i$, the nodes with value $e^{\alpha_i + x_{i,j}}$.

An example of two possible final results for the two trees, obtained for some values of $a$ and $\sigma$, is plotted in Figure 13. Notice that the higher $a^I$ (or $a^P$) is, the less nodes the respective tree have.

In order to calibrate for the parameters of Equation (33), we use a procedure inspired by [8]. The main idea is to use the discrete time version of the solution of Equation (33):

$$
X^*(t) = X^*(s) e^{-a(t-s)} + \sigma e^{-at} \int_s^t e^{au} dW_u, \quad 0 \leq s < t,
$$

which gives

$$
x(t_i) = bx(t_{i-1}) + \delta \varepsilon(t_i)
$$

with

$$
b = e^{-a \Delta t}, \quad \delta = \sigma \sqrt{1 - e^{-2a \Delta t} \over 2a}
$$

and $\varepsilon$ is a Gaussian white noise ($\varepsilon(t_i) \sim N(0, 1)$ for all $i$). Then, in order to provide the maximum likelihood estimator for the parameters $b$ and $\delta$, perform a least squares regression of the time series $x(t_i)$ on its lagged value $x(t_{i-1})$, as in Equation (35). Once we have $b$ and $\delta$, we can invert Equation (36) and derive the original parameter $a$ and $\sigma$.

References


(a) Strong mean reversion, low volatility:
\[ a^P = 3, a^I = 10, \sigma^P = 0.3, \sigma^I = 0.1 \]

(b) Small mean reversion, high volatility:
\[ a^P = 0.1, a^I = 0.1, \sigma^P = 0.7, \sigma^I = 0.2 \]

Figure 13: Trees for prices for different values of parameters. Notice that the higher \( a^P \) and \( a^I \) are, the less nodes the respective trees have: in subfigure (a) we have trees obtained with high \( a^P \), \( a^I \) and few nodes in both the trees, while in subfigure (b) we have the converse situation.


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