Modeling and valuing *make-up* clauses in gas swing contracts

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Abstract

In the last ten years, thanks to the worldwide energy liberalization process, the birth of competitive gas markets and the recent financial crisis, traditional long term swing contracts in Europe have been supplemented in a significant way by make-up clauses which allow postponing the withdrawal of gas to future years when it could be more profitable. This introduces more complexity in the pricing and optimal management of swing contracts. This paper is devoted to a proper quantitative modelization of one type of make-up clause in a gas swing contract. More in detail, we succeed in building an algorithm to price and optimally manage the make-up gas allocation among the years and the gas taking in the swing subperiods within the years: we prove that this problem has a quadratic complexity with respect to the number of years. The algorithm can be adapted to different instances of make-up clauses as well as to some forms of carry-forward clauses. Then, as an example, we show the algorithm at work on a 3-year contract and we present a sensitivity analysis of the price and of the make-up policy with respect to various parameters relative both to the price dynamics and to the swing contract. To the authors’ knowledge, this is the first time that such a quantitative treatment of make-up clauses appears in literature.

**Keywords:** swing option; price decoupling; make-up clause; dynamic programming; bang-bang controls.

**JEL Classification:** C61, C63, D81, Q49.

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1 Introduction to long term supply contracts in European gas markets

Europe is among the largest consumers of natural gas in the world, the gas being used mainly for heating and power generation. During the last thirty years natural gas has gradually replaced almost everywhere fuel oil for heating purposes and is at present competing with coal as main fuel source for electric power generation. Hence, the long term trend of natural gas demand has been historically upward sloping. The economic crisis of 2008 has strongly impacted this tendency: global gas demand fell sharply by 3% between 2008 and 2009. As reported in Table 1, the International Energy Agency (IEA) forecasts that OECD gas demand will recover slowly with consumption returning to the 2008 levels by 2012 or 2013, depending on the region.

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Table 1: OECD Natural Gas Demand by Region in billion cubic meters per year. Data for 2008-2009 are historical, for 2010 estimated, and for 2011-2013 forecasted.

Recent events concerning nuclear power generation, after the Fukushima accident, are expected to provide new strength to the long term tendency of increasing global demand for natural gas. In fact, in the medium to long term, many countries are expected to reduce their nuclear ambitions and one can reasonably expect that the natural gas will be fuel of choice to compensate for lower production of nuclear energy.

Despite its significant consumption, Europe, meant either as the OECD or the European Union (EU), has only a limited domestic production compared to its consumption and the excess demand is covered by massive natural gas imports from producer countries like Russia and Algeria, as shown in Table 2 where the Natural Gas Imports for the EU-27 countries is reported.

Natural gas imports are physically delivered via pipelines (Figure 1) or via LNG (Liquified Natural Gas) cargoes and were traditionally based on long term oil-linked swing contracts (10-30 years duration) in order to guarantee the security of supply of such an important energy commodity. In Europe long term gas contracts have been traditionally priced using oil-linked pricing formulas, while in the United States gas-

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1 Current membership of OECD: Australia, Austria, Belgium, Canada, Chile, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States.

2 EU-27: Austria, Belgium, Bulgaria, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden and the United Kingdom.
to-gas competition has historically determined most natural gas wholesale transactions. In the last ten years, thanks to the worldwide energy liberalization process and the birth of competitive gas markets, in almost all European countries these long term contracts have been supplemented by spot transactions (short term transactions), although long term deals still represent the pillar of the European gas system (see Table 3 and [20]).

The structure of long term gas agreements is quite standardized in Europe. As already said, these long term contracts are swing (also known as take-or-pay, see [19] for details) in nature, with the peculiarity that the strike price typically depends upon a basket of crude and refined oil products, which is averaged through time in order to smooth undesired volatility effects; for more details we refer the interested reader to [1, Section 3.1]. Since oil products are traded in US dollars, oil related indexes are also expressed in US dollars. Thus, typical market risk factors perceived by European importers are represented by the USD/EUR exchange risk, gas cost \( I_n \), and the price differential between import cost in Euros and local market prices \( P_n \) settled daily by gas market exchanges. We emphasize, however, that USD/EUR exchange rate volatility is comparatively low compared with typical spot gas price volatility (Figure 2(a)).
(a) TTF (Title Transfer Facility - Netherlands gas hub) and FX (EUR/USD exchange rate) volatilities. TTF volatility is 3 to 10 times larger than that of FX.

(b) CAL10 and CAL11 (calendar forward) of TTF and GR07 (ENI Gas Release 2007, gas price formula) contracts.

Figure 2: Gas volatility and price [2].

These long term contract structures and oil indexation have their origins in the early European gas market of the 1970s. Since that time sources of gas have increased, making gas markets and infrastructure much denser and open to competition. From 2008 onwards this traditional market framework has significantly changed especially with regard to the oil-to-gas price relationship. At present, the demand drop following the financial crisis with the subsequent economic recession together with the significant increase in LNG (Liquefied Natural Gas) and unconventional gas supply sources flowing to Europe have generated a substantial and quite persistent decoupling of European gas market prices from the price of oil: since 2008, European gas markets are pricing systematically and significantly below indexes usually used for the strike price I above (see Figure 2(b)), so the spread $P - I$ has became negative.

Obviously, this market phenomenon has created a panic situation for the owners of classical long term gas supply contracts. Significant losses are currently being faced by pipeline importers due to this kind of oil-to-gas price decoupling. Moreover, the structural market changes have created an increased sense of uncertainty about future development of the European gas market. Interested readers may refer to [20] for a detailed and updated analysis of oil-to-gas decoupling.

This new market scenario has induced many long term importers to engage in a renegotiation process with their suppliers and has focused their attention on the optimization and hedging possibilities which are naturally embedded in the current contracts. In fact, traditionally long term forward gas contracts are often equipped with some volumetric flexibilities which permit prompt management of year-to-year fluct-
tations in gas demand. Among those, in this market situation a new and particular importance arose for the so-called make-up and carry forward clauses, which flank traditional constraints as minimum and maximum withdrawal quantity established for every contract year and every contract sub-period (day or month). Basically, these clauses allow the buyer of the contract to delay or anticipate respectively the withdrawal of gas from one year to another within the full respect of sub-period capacity constraints. In particular, the introduction of make-up clauses has become very important for European long term contract holders: in fact, in the recent oil-to-gas price decoupling situation, contract holders were induced to delay gas delivery as much as possible for the sake of loss minimization. With a make-up clause contract holders can effectively postpone the delivery of gas when it is too expensive with respect to market prices, hoping that future gas prices will rise so that the exercise of the contract rights becomes profitable.

Furthermore, gas and oil markets are extremely volatile (see, for example, the volatility of gas market TTF (Title Transfer Facility, Netherlands gas hub) in Figure 2(a)) and consequently contracts optimization and hedging must be performed dynamically through time in order to protect the contract’s value or at least contain financial losses. The optimization/valuation problem of standard swing contracts is not a trivial problem per se, as sub-period decisions typically impact the possibility of exercising the option in the future due to annual volume constraints. Thus, in recent years swing options have received extensive treatment in the literature (see for instance [5, 19] and references therein with regard to gas markets, and references in [3, 4, 6, 13] for swing options in more general markets).

The presence of make-up clauses further complicates things and introduces more complexity. Surprisingly, the quantitative literature appears to be scarce: in the authors’ opinion, this is due to the fact that a make-up clause is worth more in a market where price decoupling is high, and the need to study such markets arose only in the last years. At a qualitative level, for instance, the make-up clause is described in [21, 23]. An algorithm to evaluate a swing contract with the carry-forward clause using the least square Monte Carlo approach is presented in [13], where the authors claim that the make-up can be evaluated similarly: this is true only for make-up clauses with a single final installment, i.e. when the make-up gas is to be paid only when it is called back (which corresponds to letting $\alpha = 0$ in Rule 4 of Section 2.3). However, typical make-up clauses have a double-installment mechanism, i.e. the make-up gas has to be paid when nominated with a price proportion $\alpha > 0$ and when called back with a price proportion $1 - \alpha$ (as an example, [21] report $\alpha \in (0.85, 1]$ as typical proportions). To the authors’ knowledge, an algorithm to properly price (and find an optimal policy for) a swing contract with such a general double-installment make-up clause and where both the market and strike price are stochastic variables has until now never been presented. The aim of this paper is exactly to fill this gap in the literature. In particular, we will describe, frame and solve the optimization issues related to the presence of make-up clauses in long term swing contracts. Finally, we use the algorithm in order to explore the value of the contract with respect to the peculiar constraints introduced by
the make-up.

The paper is organized as follows. In Sections 2.1 and 2.2 we outline the generic structure of gas swing contracts and in Section 2.3 we describe a particular instance of make-up clause. Then in Section 2.4 we frame the problem mathematically and indicate an algorithm for its solution, describing analytically both its formal representation and the various steps for the solution. In Section 2.5 we discuss the computational cost of our approach and obtain a quadratic cost with respect to the duration in years of the make-up clause. In Section 3 we extend our approach to another form of make-up clause as well as to some instances of carry-forward clause. In Section 4 we present a detailed example for a 3-year make-up clause, and in Section 5 we use this contract to perform a sensitivity analysis in order to outline the key drivers for optimization and value protection given the current gas market scenario: in order to perform this analysis we calibrate two mean-reverting trinomial models to market data (in particular, TTF for the price $P$ and ENIGR07 for the index $I$) as explained in Appendix A. Concluding remarks in Section 6 end the paper.

2 The structure of swing contracts with make-up clause

2.1 Time structure and admissible strategies

Ordinary swing contract schemes are normally defined dividing each one of the $D$ yearly delivery periods $\{[T_{j-1}, T_j]\}_{j=1,\ldots,D}$ into $N$ sub-periods $\{[t_{j,i-1}, t_{j,i}]\}_{i=1,\ldots,N}$ obtaining the sequence $\{t_{j,i}\}$ such that

$$0 = T_0 = t_{1,0} < t_{1,2} < \ldots < t_{1,N} = T_1 = t_{2,0} < t_{2,1} < \ldots$$

$$\ldots < t_{j,i} < \ldots < t_{j,N} = T_j = t_{j+1,0} < \ldots < t_{D,N} = T_D$$

In particular, in every year $[T_{j-1}, T_j]$ we have the $N + 1$ points $(t_{j,i})_{i=0,\ldots,N}$ such that $t_{j,0} = T_j$ and $t_{j,N} = T_{j+1}$.

We are also assuming that $N$ is also the number of swing rights that the holder may exercise in each year, and that these rights may be exercised exactly at the points $t_{j,i}$, for $i = 0, \ldots, N - 1$ i.e. at the beginning of every sub-period. For example if the decisions are taken month by month, at the beginning of every month, $N = 12$, if day by day $N = 365$.

Denote by $u_{j,i}$ the quantity of gas the holder decides to buy in the sub-period $[t_{j,i}, t_{j,i+1})$, $i = 0, \ldots, N - 1$, and by $z_{j,i}$ the cumulated gas quantity at time $t_{j,i}$. In particular we set $z_{j,0} = 0$ for all $j = 1, \ldots, D$ and

$$z_{j,i+1} = \sum_{k=0}^{i} u_{j,k} = z_{j,i} + u_{j,i} \quad \forall i \in \{0, \ldots, N - 1\}$$

Over each one of the $N$ sub-periods, minimum (mDQ) and maximum (MDQ) delivery quantities are established in the contract, which usually reflect physical effective
transportation capacity limitations: thus, the quantities \( u_{j,i} \) are constrained by

\[
mDQ \leq u_{j,i} \leq MDQ \quad \forall i = 0, \ldots, (N - 1), \quad \forall j = 1, \ldots, D
\]  

(2)

For every contractual year, minimum and maximum quantities are also established, called respectively the minimum annual quantity (mAQ) and the annual contract quantity (ACQ). The difference between the maximum gas that the holder could physically take and his contract right is thus given by

\[
\mathcal{M} := N \cdot MDQ - ACQ
\]

while the difference between the minimum gas that the holder must take by contract and the minimum which he could physically take is given by

\[
\mathcal{M} := mAQ - mDQ \cdot N
\]

(4)

Often we have non-trivial volume constraints, in the sense that

\[
\mathcal{M} > 0, \quad \mathcal{M} > 0
\]

(5)

Thus, in the light of the discussion above, without any additional clauses and with non-trivial constraints we have

\[
N \cdot mDQ < mAQ \leq z_{j,N} \leq ACQ < N \cdot MDQ \quad \forall j = 1, \ldots, D
\]

Penalty payments can be imposed if the volume constraints are exceeded in order to stimulate the buyer to respect the volumetric limits imposed (see for example [5]), but in this paper we do not take into account these penalties.

The difference between swing contracts with trivial and non-trivial volume constraints is extremely important in the pricing and hedging of the contract itself. In fact, with non-trivial volume constraints the holder must take into account, at time \( t_{j,i} \), not only the quantity \( u_{j,i} \) which would be optimal for that period, but also the effects of this quantity on the future decisions that he will subsequently be allowed to take. This suggests modeling the so-called space of controls, i.e. the set where \( u_{j,i} \) is allowed to take values, in the following way. For a given year \( j = 1, \ldots, D \), assume that we have a final constraint \( z_{j,N} \in [z, \bar{z}] \) for some \( 0 \leq z < \bar{z} \) (\([z, \bar{z}] = [mAQ, ACQ]\) in the absence of make-up or other clauses). Then, for a given time \( t_{j,i} \), the space of controls \( \mathcal{A}(t_{j,i}, z_{j,i}, [z, \bar{z}]) \) will in general depend on the time \( t_{j,i} \), the cumulated quantity \( z_{j,i} \) and the interval \([z, \bar{z}]\).

By the constraints (2) and construction of \( z_{j,i} \), at time \( t_{j,i} \) we can restrict our attention to the case when \( z_{j,i} \) satisfies the constraints

\[
mDQ \cdot i \leq z_{j,i} \leq MDQ \cdot i \quad \forall i = 0, \ldots, N
\]

and

\[
N \cdot mDQ \leq z \leq \bar{z} \leq N \cdot MDQ
\]
Figure 3 shows an example of the admissible area. In this case, we are not allowed to take 
\(u_{j,i} = \text{MDQ}\) for all \(i = 0, \ldots, N - 1\): in fact, there exists a time \(\tau_1\) such that, if we have always taken this minimum for \(t \leq \tau_1\), then for \(t > \tau_1\) we have to switch to \(u_{j,i} = \text{MDQ}\) in order to reach \(z\). This point \(\tau_1\) is the common point between the two lines \(z = \text{MDQ}(t - t_{j,0})\) and \(z = \text{MDQ}(t - t_{j,N} + \Delta t)\), \(\forall t \in [t_{j,0}, t_{j,N}]\). A simple calculation leads to

\[z_{j,i} \geq r_{\min}(t_{j,i}, z) = \max\{\text{MDQ}(t_{j,i} - t_{j,0}), \text{MDQ}(t_{j,i} - t_{j,N}) + z\}\]

Similarly, we are not allowed to take always \(u_{j,i} = \text{MDQ}\) either: in fact, there exists a time \(\tau_2\) such that, if we have always taken this maximum for \(t \leq \tau_2\), then for \(t > \tau_2\) we have to switch to \(u_{j,i} = \text{MDQ}\) in order to reach, and not exceed, \(z\). The boundary for \(z_{j,i}\) in this case is

\[z_{j,i} \leq r_{\max}(t_{j,i}, z) = \min\{\text{MDQ}(t_{j,i} - t_{j,0}), \text{MDQ}(t_{j,i} - t_{j,N}) + z\}\]

Figure 3 shows an example of the admissible area.

In conclusion, the correct form of the space of controls \(A(t_{j,i}, z, [z, \bar{z}])\) at time \(t_{j,i}\), given the constraint \(z_{j,N} \in [\underline{z}, \overline{z}]\) and the cumulated quantity \(z_{j,i} = z\), is given by

\[A(t_{j,i}, z, [\underline{z}, \overline{z}]) := \{u_{j,i} \in [\text{MDQ}, \text{MDQ}] \mid z + u_{j,i} \in [r_{\min}(t_{j,i+1}, z), r_{\max}(t_{j,i+1}, \bar{z})]\}\]

which appears implicitly in [3, Equation 7] and is also a discretized version of the one in [6].

2.2 The price of a standard swing contract

We now present a standard procedure to price a swing option without the presence of additional clauses (such as make-up).

Let \(P_{j,i}\) and \(I_{j,i}\) be respectively the prices of gas and index in year \(j = 1, \ldots, D\), sub-period \(t_{j,i}, t_{j,i+1}\), \(i = 0, \ldots, N - 1\): the contract holder has to buy the gas at the price \(I_{j,i}\) and can sell it at the price \(P_{j,i}\): of course with this notation we have \((P_{j,N}, I_{j,N}) = (P_{j+1,0}, I_{j+1,0})\) for each year \(j = 1, \ldots, D - 1\). Although \(I = (I_{j,i})_{j=1, \ldots, D, i=0, \ldots, N}\) is a time average on several subperiods and thus has relevant memory effects, as mentioned in the Introduction, in the following we make the simplifying assumption that \((P_{j,i}, I_{j,i})_{j=1, \ldots, D, i=0, \ldots, N}\) evolve as a two-dimensional Markov process under a pricing measure \(\mathbb{P}\), which is used in all the mathematical expectations that follow, while the numerical implementation that we use for the analysis of the next section will make use of the particular specification that we describe in Appendix [A]. We also assume that \(\mathbb{E}[e^{-r t_j}, P_{j,i} \mid F_{j,i'}] = F_{j,i}^P < +\infty\) and \(\mathbb{E}[e^{-r t_j}, I_{j,i} \mid F_{j,i'}] = F_{j,i}^I < +\infty\), where \(F_{j,i}^P, F_{j,i}^I\) represent the forward prices of \(P\) and of \(I\), respectively, for the delivery time \(t_{j,i}\).
Figure 3: Typical admissible area for one year. Here $\bar{z} < \underline{z}$, leaving some optionality for the total intake $z_{j,N}$. If $\bar{z} = \underline{z}$ (typical of years when some make-up gas is nominated or called back), we have the constraint $z_{j,N} = \bar{z} = \underline{z}$ and the admissible region is like those in Figure 4.

The objective of the contract holder is to maximize the discounted global margin of the contract (i.e. minimize the total loss), i.e., (s)he wants to calculate the value of

\[
V(0, p_{1,0}, t_{1,0}, 0) = \sup_{u \in A} \mathbb{E} \left[ \sum_{j=1}^{D} \sum_{i=0}^{N-1} e^{-rt_{j,i}} u_{j,i} (P_{j,i} - I_{j,i}) \right]
\]

where the set $A$ of admissible controls is defined by

\[
A := \{(u_{j,i})_{j,i} \text{ adapted to } (P_{j,i}, I_{j,i})_{j,i} \text{ and s.t. } u_{j,i} \in A(t_{j,i}, z_{j,i}, [mAQ, ACQ])\}
\]

in the absence of a make-up clause, and $r \geq 0$ is the risk-free annual interest rate. Equation (7) follows from the fact that $z_{j,0} = 0$ for all $j = 1, \ldots, D$, i.e. in the absence of a make-up clause the swing contract can be valued independently year by year.
It is a standard result (see e.g. \[3, 5, 19\]), and it will also follow as a particular case of our results in Section 2.4, that this maximisation problem can be solved by the use of the Dynamic Programming: for each year $j = 1, \ldots, D$, define the deterministic functions

$$
V_j(N, p, \lambda, z) := 0, \quad \text{(8)}
$$

$$
V_j(i, p, \lambda, z) := \max_{u \in \mathcal{A}(t_j, i, [mAQ, ACQ])} E_{P_j} \left[ e^{-r t_j} u (p - \lambda) + V_j(i + 1, P_j, i + 1, I_j, z + u) \right] \quad \forall i < N \quad \text{(9)}
$$

where $E_{P_j}$ indicates the expectation conditional to $P_j = p$ and $I_j = \lambda$ (recall that, as these are Markov processes, these values are a sufficient statistics for the whole information up to subperiod $i$ of year $j$). Then the original problem in Equation (7) is reduced to calculating

$$
V(0, p_1, \lambda_1, 0, 0) = E \left[ \sum_{j=1}^{D} V_j(0, P_j, I_j, 0, 0) \right]
$$

2.3 Modeling the make-up clause

This subsection is devoted to the analytical representation of the make-up clause and its constraints. While long term contracts may have a length of 10-30 years, make-up clauses are typically written on a limited period of the contract life, often from 3 to 5 years. As explained in Section 2.2, a contract without make-up clause can be evaluated as the sum of mutually independent yearly contracts; thus, we can split a contract with make-up written only on a subperiod of the whole contract life in two parts: the first part is a swing contract with a make-up clause with length equal to the original make-up clause, while the other part covers all the years when the make-up is not written. Thus, without loss of generality we can assume that the make-up clause is written on the whole contract’s length, $D$ years.

For each year $j = 1, \ldots, D$, call $M_j$ the make-up gas nominated and $U_j$ the make-up gas called back in year $j$. With this notation, we assume that the precise structure of the make-up clause follows these rules.

1. For each year $j = 1, \ldots, D - 1$, the contract holder is allowed to take $z_{j,N} < mAQ$, provided $u_{j,i} \geq MDQ$ for all $i = 0, \ldots, N - 1$.

Thus, the make-up gas nominated in year $j$ is

$$
M_j := (mAQ - z_{j,N})^+ \quad \text{and must satisfy} \quad M_j \in [0, M], \quad \text{(10)}
$$

where $x^+ := \max(x, 0)$ and $M$, defined in Equation (4), is also the maximum quantity of make-up gas that can be physically nominated in a given year.

2. The make-up $M_j$ nominated in year $j$ can be called back in one or more subsequent years (the quantity $M_j$ can be split and called back in more than one year). This is possible only if the $ACQ$ quantity has been reached in that year, and of course in that year one must still have $u_{j,i} \leq MDQ$ for all $i = 1, \ldots, N$. 


Thus, the make-up gas called back in year $j = 2, \ldots, D$ is

$$U_j := (z_{j,N} - ACQ)^+$$

and is such that $U_j \in [0, \overline{M}]$ \hfill (11)

where $\overline{M}$, defined in Equation (3), is also the maximum quantity of make-up gas that can physically be called back in a given year.

3. It is not possible to call back make-up gas before having nominated it, and at year $D$ all the nominated make-up gas must have been called back.

Thus, if we define the cumulated gas debt at year $j$, i.e. the make-up gas not yet called back, as

$$M_j = \sum_{k=1}^{j} M_k - \sum_{k=2}^{j} U_k = \sum_{k=1}^{j} (M_k - U_k),$$

then $U_1 = M_D = 0$, $M_j \geq 0$ for all $j = 1, \ldots, D - 1$ and $M_D = 0$. Moreover,

$$M_{j+1} = M_j + M_j - U_j = M_j + (mAQ - z_{j,N})^+ - (z_{j,N} - ACQ)^+$$

Notice that conditions 2. and 3. imply, for example, that if at the beginning of the last contract year of the make-up clause we have some make-up gas not called back, i.e. $M_{D-1} > 0$, in year $d$ we necessarily have to reach the quantity $ACQ + M_{D-1}$.

Remark 1 More in general, for all years $j = 1, \ldots, D$, the definition of $M_j$ implies that $M_j \leq j \cdot \overline{M}$ and $M_j \leq (D - j) \cdot \overline{M}$. By combining these two constraints, the maximum gas debt is possible at year

$$\bar{j} := \frac{D \overline{M}}{\overline{M} + \overline{M}}$$

if $\bar{j}$ is integer, and at one of the two nearest years if $\bar{j}$ is not integer. In particular, the gas debt $M_j$ can increase without constraints for $j < \bar{j}$ and must possibly be decreased for $j > \bar{j}$.

4. The price of the make-up quantity nominated in year $j$ and called back in year $k$, subperiod $i$, is defined as the weighted sum of two components respectively paid at two different times:

a) at time $t_{j,N}$ (i.e. at the end of year $j$ when $M_j$ becomes known) the buyer pays the make-up gas at the price $\alpha \Gamma_j$ for some $\alpha \in (0, 1)$ defined in the contract, where $\Gamma_j$ is the average index price observed in year $j$;

b) at time of withdrawal $t_{k,i}$, the price paid is $(1 - \alpha) I_{k,i}$.

The price of make-up gas, as defined above, is associated to the gas volume $u_{k,i}$ physically delivered at time $t_{k,i}$. This means that the part $\alpha \Gamma_j$ in (a) of the price needs
to be capitalized from time $T_j = t_{j,N}$ up to time $t_{k,i}$: thus, the price $\mathcal{I}_{j,k,i}$ at time $t_{k,i}$ of the make-up gas nominated in year $j$ and called back in year $k$, subperiod $i$ is

$$\mathcal{I}_{j,k,i} = \alpha \Gamma_j e^{r(t_{k,i} - t_{j,N})} + (1 - \alpha) I_{k,i}$$

(14)

By discounting at time $T_0 = 0$ the price of make-up gas called back at time $t_{k,i}$, we have

$$e^{-r_{k,i}}\mathcal{I}_{j,k,i} = \alpha \Gamma_j e^{-r_{j,N}} + (1 - \alpha) I_{k,i} e^{-r_{k,i}}$$

It follows that, in a year $j = 1, \ldots, D$, in which the make-up clause is exercised to nominate or call back gas, the residual value of the swing contract at subperiod $i = 0, \ldots, N - 1$ for that year with the control policy $u_j := (u_{j,i})_{i=0,\ldots,N-1}$ is given by

$$J_j(i,p,\nu,\omega; u_j) := \mathbb{E}^{p,j} \left[ \sum_{k=i}^{N-1} e^{-r_{j,k}} u_{j,k} (P_{j,k} - A_\alpha(u_{j,k},\omega_j) I_{j,k}) - e^{-r_{j,N}} \alpha \Gamma_j M_j \right]$$

(15)

where

$$A_\alpha(u,\omega) := 1 - \alpha \left( 1 - \frac{ACQ - \omega}{u} \right) ^+ 1_{\{ACQ-MDQ<\omega\leq ACQ\}} - \alpha 1_{\{\omega>ACQ\}}$$

is a pricing coefficient in the interval $[1 - \alpha, 1]$ for $I_{j,k}$ to accommodate the gas quality (ordinary below the ACQ, called back from previous years above). Here, the main apparent difference with respect to the case when no make-up is exercised is that we can end up a year with a non-null position in the make-up gas, i.e. with $M_j - U_j \neq 0$ (notice that this notation is not ambiguous as $M_j$ and $U_j$ cannot be both different from zero), where the quantity $M_j - U_j$ is by definition a deterministic function of $z_{j,N}$, thus of $u_j$. Notice that this generalizes the payoff to be maximised in Equation (7), which is reobtained for $z_{j,N} \in [mAQ, ACQ]$, i.e. $M_j = U_j = 0$, and setting $z_{j,0} = 0$.

As a result, the total value of the swing option with the make-up clause described above is given by

$$\sup_{u \in \mathcal{A}} \mathbb{E} \left[ \sum_{j=1}^{D} J_j(0, p_{j,0}, \nu_{j,0}, 0; u_j) \right]$$

(16)

where $u = (u_{j,i})_{j=1,\ldots,D, i=0,\ldots,N-1}$ must now belong to the set

$$\mathcal{A} := \left\{ (u_{j,i})_{j,i} \text{ adapted to } (P_{j,i}, I_{j,i})_{j,i} \mid u_{j,i} \in [mDQ, MDQ], \right\}$$

$$U_1 = M_D = 0, M_j \geq 0 \forall j = 1, \ldots, D, M_D = 0$$

and the functions $J_j$ are given by Equation (15). Here, the constraints on $U$ and $M$ induce constraints on $(u_{j,i})_{j,i}$, which will be treated in detail in the next subsection.
2.4 The price of swing contracts with make-up clauses

We have just seen that the set of admissible strategies for \((u_{ji})_j\) for a given year \(j = 1, \ldots, D\) now depends on the gas debt \(M_{j-1}\) arriving from the previous years. For this reason, this quantity has to be explicitly taken into account in the evaluation of the swing contract for that year. More in detail, now we define the value function as

\[
V_j(i, p, \iota, z, M_{j-1}) := \sup_{u \in A} \mathbb{E}_{p,\iota}^j \left[ J_j(i, p, \iota, z; u_j) + \sum_{k=j+1}^D J_k(0, P_k, 0, I_k, 0; u_k) \right]
\]  

We now build a Dynamic Programming algorithm as in [5, 19]: for each year \(j = 1, \ldots, D\), define the deterministic functions:

- if \(j = D\), then \(M_D = 0\) and \(U_D = M_{D-1}\) (recall that \(M_D = 0\)), so we let \(z := mAQ\) if \(M_{D-1} = 0\) and \(\overline{z} := ACQ + M_{D-1}\) (18)

and define

\[
V_j(N, p, \iota, z, M_{j-1}) := 0, \quad V_j(i, p, \iota, z, M_{j-1}) := \max_{u \in A} \mathbb{E}_{p,\iota}^j \left[ e^{-r \iota} u(p - A_\alpha(u, z) \iota) + V_j(i + 1, P_{j+1}, I_{j+1}, z + u, M_{j-1}) \right]
\]

Notice that the functions \(V_j\) depend on \(M_{j-1}\) through \([z; \overline{z}]\).

- for \(j = 1, \ldots, D - 1\) the key quantity is \(M_{j-1}\) which is known at the beginning of the year. Assume that \(M_{j-1} \leq (D - j + 1)\overline{M}\). For the lower bound we have two cases.

  - If \(M_{j-1}\) is admissible also for year \(j\), i.e. if \(M_{j-1} \leq (D - j)\overline{M}\), then we can nominate some other make-up gas \(M_j\) as long as

    \[
    M_j = M_{j-1} + M_j \leq (D - j)\overline{M} \Rightarrow M_j \leq (D - j)\overline{M} - M_{j-1}
    \]

    Taking into account the mdQ constraints, the lower bound for \(z_{j,N}\) is:

    \[
    \overline{z} := mAQ - \min \{(D - j)\overline{M} - M_{j-1}, \overline{M}\}
    \]  

  - If \(M_{j-1}\) is not admissible for year \(j\), i.e. if \(M_{j-1} > (D - j)\overline{M}\), then we must call back some make-up gas in order to obtain a final cumulated quantity \(M_j\) admissible for year \(j\), i.e.

    \[
    M_j = M_{j-1} - U_j \leq (D - j)\overline{M} \Rightarrow U_j \geq M_{j-1} - (D - j)\overline{M}
    \]

    So the lower bound for \(z_{j,N}\) is now

    \[
    \overline{z} := ACQ + M_{j-1} - (D - j)\overline{M}
    \]
For the upper bound $\overline{z}$, we do not need to distinguish between the two previous cases and let

$$\overline{z} := ACQ + \min \{ M_{j-1}, \overline{M} \}$$  \hspace{1cm} (23)$$

Finally, define

$$V_j(N, p, t, z, M_{j-1}) := V_{j+1}(0, p, t, 0, M_{j-1} + (mAQ - z)^+ - (z - ACQ)^+) + e^{-rt}j_{N} \Gamma_j(mAQ - z)^+,$$  \hspace{1cm} (24)$$

and for $i = N - 1, \ldots, 0$ define $V_j(i, \cdot, \cdot, \cdot)$ exactly as in Equation (20).

**Theorem 2.1**

1. The deterministic functions $V_j(\cdot, \cdot, \cdot, \cdot)$, defined by the dynamic programming equations (19), (20) and (24) are such that $V_1(0, P_{1,0}, I_{1,0}, 0, 0)$ coincides with the value of the swing option with the make-up clause in Equation (16).

2. There exists an optimal Markovian consumption $u_{j,i}^* = u(t_{j,i}, P_{j,i}, I_{j,i}, z_{j,i}, M_{j-1})$, where $u(\cdot, \cdot, \cdot, \cdot, \cdot)$ is given by the maximum argument in the dynamic programming equation (20).

3. If the quantities

$$K := \frac{M}{MDQ - mDQ} \quad \text{and} \quad \overline{K} := \frac{\overline{M}}{MDQ - mDQ},$$  \hspace{1cm} (25)$$

are integers, then there exists an optimal bang-bang Markovian consumption $u_{j,i}^* = mDQ$ or $u_{j,i}^* = MDQ$ for all $j = 1, \ldots, D$, $i = 0, \ldots, N - 1$. Moreover, $M_j$ turns out to be an integer multiple of $MDQ - mDQ$ for all $j = 1, \ldots, d$.

**Proof.** We proceed in analogy with [5] and [4].

1. As $0 \leq t_{j,i} \leq T$, $0 \leq z \leq N \cdot MDQ$, $M \leq D \cdot M$ and $E[P_{j,i}] = F^{P}_{j,i} < +\infty$, $E[I_{j,i}] = F^{I}_{j,i} < +\infty$, then the assumptions $(F^+, F^-)$ in [7, Proposition 8.5] are satisfied, so the argument follows.

2. The right-hand side of Equation (20) is continuous in $u$ and $A(t_{j,i}, z, [\xi, \overline{z}])$ is a compact set contained in $[mDQ, MDQ]$, thus the maximum is attained for $u \in A(t_{j,i}, z, [\xi, \overline{z}])$ again by applying [7, Proposition 8.5].

3. As in [4], it can be proved that the functions $V_j(\cdot, \cdot, \cdot, \cdot)$, $j = 1, \ldots, D$, $i = 0, \ldots, N - 1$ are continuous and concave on $z$ and piecewise affine on the intervals

$$[k \cdot mDQ + (i - k) \cdot MDQ, (k + 1) \cdot mDQ + (i - k - 1) \cdot MDQ], \quad k = 1, \ldots, i.$$  \hspace{1cm} (26)$$

We now prove the claim by induction on $j = 1, \ldots, D$. If $j = 1$, then $M_0 = 0$ by definition. For a given year $j = 1, \ldots, D$, assume for now that $M_{j-1}$ is an integer multiple of $MDQ - mDQ$. Then this, together with the condition $K, \overline{K} \in \mathbb{N}$ ensures that $[r_{\min}(t_{j,i}, z), r_{\max}(t_{j,i}, \overline{z})]$ is exactly the union of suitable intervals of the kind of Equation (26). Thus, if $z = k \cdot mDQ + (i - k) \cdot MDQ$ for some $k = 0, \ldots, i$, then
the function to be maximised in Equation (20) is affine on \( u \), thus its maximum point is \( u^*_{j,i} = mDQ \) or \( u^*_{j,i} = MDQ \). It can then be proved by induction that, since \( z_{j,0} = 0 \), the optimal \( u^*_{j,i} \) is such that \( z_{j,i} = k \cdot mDQ + (i - k) \cdot MDQ \) for some \( k = 0, \ldots, i \): this also implies that \( M_j \) too will be an integer multiple of \( MDQ - mDQ \), and the conclusion follows.

\[ \square \]

**Remark 2** Part 3. of the theorem above is essentially a consequence of the linear structure of the payoff function in Equations (15–16): the result is that in every year \( j \), subperiod \( i \), the optimal quantity \( u_{j,i} \) can be safely chosen to be either the maximum (MDQ) or the minimum (mDQ) admissible for that substep. This kind of control is called of bang-bang type, and it was already found in [5] with smoother payoffs, and studied in deep detail in [4]. Qualitatively, this is due to the fact that, if the withdrawal is profitable in the subperiod, then the better choice is the maximum quantity we can take; conversely, if the withdrawal is not profitable, then the better choice is to take the minimum quantity we can.

### 2.5 Computational cost

As we have seen, the pricing problem for a swing option with make-up clause boils down to the maximization problem of Equation (17). Unfortunately, this maximization cannot be carried out by analytic means, as a closed form for \( V_j \) is not known even in the simplest case of a standard swing option without make-up clause. Thus, this maximization must be carried out via numerical methods.

The most efficient way to do this is to assume that the quantities \( K \) and \( K \) in Equation (25) are integers, so that the results of Theorem 2.1 hold. This induces a quantization in the candidate optimal make-up gas debt \( (M_j)_{j=1,\ldots,D} \): in fact, since this process at optimality has values which are multiple integers of \( MDQ - mDQ \), we obtain that the resulting candidate optimal quantities for \( M_j, j = 1, \ldots, D \), are finite in number. More in detail, the sequence \( (M_j)_{j=1,\ldots,D} \) is bound to have a finite number of nonnegative values in each year \( j = 1, \ldots, D - 1 \), with \( -M \leq M_j - M_{j-1} \leq M \), i.e. the increments can have at most \( K + K + 1 \) distinct values, corresponding respectively to the cases when \( M_j > 0, U_j > 0 \) and \( M_j = U_j = 0 \).

With this in mind, we can calculate the computational cost needed to price a \( D \)-year swing option with make-up clause, via the same backward recursion used in the Dynamic Programming algorithm seen in Section 2.4. In the \( D \)-th year, we can start with \( M_{D-1} \) taking at most \( K + 1 \) different values, each one of these leading to a different optimization problem: since also the values of \( (z_{j,i})_{j,i} \) are quantized via the bang-bang optimal process \( (u_{j,i})_{j,i} \) for each of these optimization problems we have a total of \( O(N^2) \) states which can be assumed by \( (z_{j,i})_{j,i} \) at optimality. Having solved the \( K + 1 \)

\[ ^3 \text{precisely} \leq \frac{N(N+1)}{2} \], which is the number of nodes of a complete recombining binomial tree.
problems for the last year, we can attach the value functions thus obtained to the terminal nodes of year $D - 1$: notice that also in this case, as now $M_{D-2}$ can assume at most $2\bar{K} + 1$ distinct values, we will have to model and solve at most $2\bar{K} + 1$ distinct optimization problem, each one having as terminal condition the values of the $\bar{K} + 1$ problems of year $D$. These numbers do not multiply, because once we have obtained the values for the $\bar{K} + 1$ problems of year $D$ for each possible starting state $(P_{D-1,N}, I_{D-1,N}) = (P_{D,0}, I_{D,0})$, we can take these values as terminal values to use in the computation for year $D - 1$.

In this spirit, we are now ready for a result on the computational cost of the pricing of a swing option with make-up clause.

**Theorem 2.2** If the quantities $K$ and $\bar{K}$, defined in Equation (25), are integer, then the order of distinct subproblems to be solved is $O(N^2 D^2)$.

**Proof.** First of all consider the 2-dimensional process $(j, M_j)_{j=1,...,D}$: then the distinct states that this process can assume, at optimality, are in 1-1 correspondence with the integer solutions $(x,y)$ of the system

\[
\begin{cases}
  x \geq 0, \\
  x \leq \bar{K}(D - y), \\
  x \leq Ky,
\end{cases}
\]

In fact, if $(x,y)$ is such a solution, then $M_x = (MDQ - mDQ)y$ is a possible value, at time $x$, of an optimal path for $M$ by Theorem 2.1. Conversely, by the same theorem, if $K$ and $\bar{K}$ are integer then $M_x$ is an integer multiple of $MDQ - mDQ$. Since for each of these possible states we must solve a separate optimization problem for the corresponding year, the number of optimization subproblems for all the values of $(z_{j,i})_{j,i}$ are of order $O(N^2)$, and their total number is the sum of these, the proof boils down to find the total number $N$ of integer solutions of the system (27). By recalling the definition of $\bar{j}$ in Equation (13) and the discussion below, first of all we rewrite $\bar{j}$ as

\[
\bar{j} = \frac{D\bar{K}}{K + \bar{K}}.
\]

Now, the region of the solutions of the system (27) is the union of the two triangular regions $\{(x,y) \in \mathbb{N}^2 | x \geq 0, x \leq Ky, y \leq \bar{j}\}$ and $\{(x,y) \in \mathbb{N}^2 | x \geq 0, x \leq \bar{K}(D - y), y > \bar{j}\}$. It is then easy to see that

\[
N = \sum_{\ell=0}^{\bar{j}} (1 + K\ell) + \sum_{\ell=\bar{j}+1}^{D} (1 + K(D - \ell)) = D + 1 + K\sum_{\ell=1}^{\bar{j}} \ell + \bar{K} \sum_{\ell=\bar{j}+1}^{D-1} (D - \ell) =
\]

\[
= D + 1 + K \frac{[\bar{j}] \cdot [\bar{j} + 1]}{2} + \bar{K} \frac{[D - \bar{j} - 1] \cdot [D - \bar{j}]}{2}
\]

where $[x]$ denotes the integer part of $x$. By noticing that for all $x > 0$ we have $D[x] \leq [Dx]$ and that

\[
K \cdot \bar{j} = \frac{D\bar{K}}{K + \bar{K}} = \bar{K}(D - \bar{j})
\]

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then we have
\[
N \leq D + 1 + \frac{1}{2} \left[ \frac{DK\overline{K}}{K + \overline{K}} \right] ([\overline{j} + 1] + [D - \overline{j} - 1]) \leq \frac{1}{2} \frac{K\overline{K}}{K + \overline{K}} D^2 + D + 1
\]
i.e. \( N = O(D^2) \). By multiplying this for \( O(N^2) \) (the number of subproblems for given year \( j = 1, \ldots, D \) and state of make-up debt \( M_j \), we obtain that the computational cost is of order \( O(N^2D^2) \), as desired

We show in Figure 4 an illustration of these numbers for \( D = 2, 3, 4 \), by making the simplifying assumption that \( \overline{K} = K =: K \).

![Diagram](image)

Figure 4: In subfigure (a), we can only obtain \( M_1 (= M_1 = U_2) \) among \( K + 1 \) distinct values, and the corresponding value function is then used in the final values of the optimization problem of year 1, so the total number of optimization problems to be solved is \( K + 1 \). In subfigure (b) we can obtain \( M_2 (= U_3) \) among \( K + 1 \) distinct values and then \( M_1 (= M_1) \) among \( K + 1 \) distinct values, so the number of optimization problems to be solved in sequence is \( 2K + 2 \). In subfigure (c) we can obtain \( M_3 (= U_4) \) among \( K + 1 \) different values, \( M_2 \) among \( 2K + 1 \) different values, and finally \( M_1 (= M_1) \) among \( K + 1 \) distinct values: so the number of optimization problems to be solved in sequence is now \( 4K + 3 \).

We also present a numerical test which validates our result of a quadratic cost in the number of years. By taking the same parameters as in Section 5, we implemented the method on a Intel i7 workstation at 3.4GHz with 8GB RAM, with the following execution times.
Table 4: Execution times, in seconds, on a Intel i7 workstation at 3.4GHz with 8GB RAM. Notice that with 1 year there is no possibility *de facto* to exercise the make-up clause.

## 3 Extensions of the model

This section is devoted to show some extensions of the algorithm proposed so far in two directions. The first one is towards another form of make-up clause, while the second one is directed to a possible application to carry-forward clauses.

### 3.1 Another form of make-up clause

While different contracts may have several slightly different definitions of the make-up clause, to the authors’ knowledge the most frequently negotiated variant to the make-up clause presented in Section 2.3 is that obtained as in [21] by modifying Rule 2 as follows: the make-up gas must be called back as soon as \( mAQ \) has been reached instead of after having exceeded \( ACQ \). In this case every time that at the beginning of the year the cumulated make-up quantity \( M_{j-1} \) is positive then all the quantity exceeding \( mAQ \) is considered to be make-up gas called back.

The maximum make-up quantity we are allowed to call back every year is now

\[
\overline{M} = ACQ - mAQ
\]

and again we can call it back only if \( M_{j-1} > 0 \).

The algorithm for this specification of the make-up clause is similar to the one presented in Section 2.4, provided we redefine some of the quantities. Precisely, redefine

\[
U_j := 1_{\{M_{j-1}>0\}} (z_{j,N} - mAQ)
\]

which must be such that \( U_j \in [0, \overline{M}] \), where the maximum make-up quantity which can be called back in a given year is now defined as \( \overline{M} := ACQ - mAQ \). This of course modifies the payoff in Equation (15), which now becomes

\[
J_j(i,p,ι,z_j,i; u_j) := \mathbb{E}_{j,i}^{p,ι} \left[ \sum_{k=i}^{N-1} e^{-r(t_j,k)u_j,k} \left( P_{j,k} - \bar{A}_\alpha(u_{j,k}, z_{j,k})I_{j,k} \right) - e^{-r(t_j,N)\alpha\bar{\Gamma}_j M_j} \right]
\]

where

\[
\bar{A}_\alpha(u, z) := 1 - \alpha \left( 1 - \frac{mAQ - z}{u} \right)^+ \mathbb{1}_{\{mAQ - MDQ < z \leq mAQ, M_{j-1} > 0\}} - \alpha \mathbb{1}_{\{z > mAQ, M_{j-1} > 0\}}
\]

is a coefficient in \([1 - \alpha, 1]\) analogous to \( A_\alpha \) of Section 2.3.
The Dynamic Programming algorithm has to be modified as follows. Replace Equation (18) with
\[ z := mAQ + M_{D-1}, \quad \tilde{z} := ACQ1_{\{M_{D-1}=0\}} + (mAQ + M_{D-1})1_{\{M_{D-1}>0\}}, \]
Equation (20) with
\[ V_j(i, p, t, z, M_{j-1}) := \max_{u \in A(t_j, z, \tilde{z})} \mathbb{E}_{j,i}^{P,t} \left[ e^{-r t_j} u(p - \bar{A}_\alpha(u, z) t) + V_j(i + 1, P_{j,i+1}, I_{j,i+1}, z + u, M_{j-1}) \right] \],
Equation (22) with \( \tilde{z} := mAQ + M_{j-1} - (D - j)M_j \), Equation (23) with \( \bar{z} := ACQ \), and finally Equation (24) with
\[ V_j(N, p, t, z, M_{j-1}) := V_{j+1}(0, p, t, 0, M_{j-1} + (mAQ - z)^+ - (z - mAQ)^+ 1_{\{M_{j-1}>0\}}) + \]
\[ + e^{-r t_{j,N}} \alpha \Gamma_j(mAQ - z)^+. \]

With these substitutions, Theorem 2.1 still applies to this case.

3.2 Carry forward clauses

The approach used in this paper can also price another clause related to swing contracts, namely some instances of the carry-forward clause as described in [13]. In general, the carry-forward (CF) right gives the holder of the option the possibility to reduce \( mAQ \) (up to a contractual amount \( ACF \) called annual carry-forward) in one year if in at least one of the previous \( d \) years the total volume taken was above \( mAQ \), while the maximum quantity which can be taken every year remains \( ACQ \).

With a slight modification of the algorithm presented in this paper we are able to price two particular instances of the carry-forward clause presented in [13], namely when the CF rights must be used in the following year (corresponding to \( d = 1 \), and when CF rights do not have a deadline, this clause being called unlimited duration carry-forward (UDCF): in this latter case there is a maximum on the CF rights one can obtain in a given year, given by \( \nu \cdot ACF \), where \( \nu > 1 \) is a contractual constant.

We now present how to price a CF clause with \( d = 1 \). First of all, define the gas credit \( U_j \) at year \( j = 1, \ldots, D \) as
\[ U_j := (z_{j,12} - mAQ)^+ \]
and \( U_0 := 0 \), and the minimum and maximum cumulated quantities for year \( j = 1, \ldots, D \) as
\[ \underline{z} := mAQ - \min(U_{j-1}, ACF), \quad \bar{z} := ACQ \] (28)
Then the Dynamic Programming algorithm can be built as follows: for the last year $D$
define
\[
V_D(N,p,\iota,z,U_{D-1}) := 0,
\]
while for $j = 1, \ldots, D-1$ define
\[
V_j(N,p,\iota,z,U_{j-1}) := V_{j+1}(0,p,\iota,0,(z - mAQ)^+),
\]
and for $i = N - 1, \ldots, 0$ define $V_j(i, \cdot, \cdot, \cdot, \cdot, \cdot)$ exactly as in Equation (30).
Then $V_1(0, p, \iota, 0, 0)$ gives exactly the value of the CF contract with $d = 1$.

We now present how to price a UDCF contract. First of all, redefine the cumulated
gas credit $U_j$ at year $j = 1, \ldots, D$ as
\[
U_j := U_{j-1} + \min(\nu \cdot ACF, z_{j,12} - mAQ)
\]
with $U_0 := 0$; with this new definition, the gas credit $U_j$ at the end of year $j$ increases
only if we exceed mAQ and decreases every time we do not reach mAQ. Redefine also
the minimum and maximum cumulated quantities for year $j = 1, \ldots, D$ as in Equation (28).

Then the Dynamic Programming algorithm can be built as follows: for the last year $D$
define the value function $V_D$ exactly as in Equations (29–30), while for $j = 1, \ldots, D-1$
define
\[
V_j(N,p,\iota,z,U_{j-1}) := V_{j+1}(0,p,\iota,0,U_{j-1} + \min(\nu \cdot ACF, z_{j,12} - mAQ)),
\]
and for $i = N - 1, \ldots, 0$ define $V_j(i, \cdot, \cdot, \cdot, \cdot, \cdot)$ exactly as in Equation (30).
Then $V_1(0, p, \iota, 0, 0)$ now gives exactly the value of the UDCF contract.

**Remark 3** In principle, it would be possible to price also CF clauses with $d > 1$ by
introducing other state variables, but this goes beyond the scope of this paper, which
is focused on make-up clauses. Moreover, in this case the computational burden could
be reduced by modelling the difference process $P - I = (P_{j,i} - I_{j,i})_{j,i}$ as a single state
variable as in [13]. Unfortunately, this is not possible with the make-up clause because
of the delayed payment structure of the make-up gas.

### 3.3 Another possible approach

The Dynamic Programming approach algorithm is not the only algorithm one can em-
ploy to solve this problem. Another approach would be the Least Square Monte Carlo
(LSMC) method as presented in [13].
This method is based on the intuition that conditional expectations in the pricing algorithm can be replaced by the orthogonal projection on some space generated by a finite set of functions, obtained using Monte Carlo simulations and least-squares regressions to estimate the orthogonal projection numerically. While the LSMC algorithm is very flexible, it may, on the other hand, be influenced by many user’s choices which are capable of influencing the pricing procedure, for instance, choices regarding the type and the number of basis functions and the number of Monte Carlo simulations used. These choices can be critical, as [22] shows that while for some type of derivatives (such as the American put) the LSMC approach is very robust, for more complex derivatives the number and the type of basis functions can slightly affect option prices. We are not stating that one algorithm is absolutely better than the other. In fact, for some type of very complex control problems LSMC algorithm represents probably the best possible solution one can pursue. Conversely, for other problems such as the swing contract with monthly granularity presented so far, where also the lattice approach works well, the decision between the two algorithm becomes a matter of what one decides to simplify.

4 Three year example

In the following we describe and analyze the case of a three year contract, i.e. the case $D = 3$ and $N = 12$. Although, as seen in Subsection 2.5, the complexity of the different kinds of control problems to be solved grows quadratically with $D$, we have decided to present a three year contract as an example since the distinct qualitative combinations of years where we can nominate and/or call back make-up gas, as $D$ grows, lead to more and more intricate combinatorical considerations, which would distract attention from our modelization. Thus, while a 2-year contract is not a very interesting example from a modellistic point of view, as at the end of the first year the make-up quantity is known and is exactly the quantity called back in the second year, on the other hand we think that $D = 3$ gives the right compromise between being able to follow exactly what goes on in the different years and the extent of the combinatorial problem.

While a longer duration of the make-up clause would not be a computational problem under our approach (see Table 4 above), we must remark that typical make-up clauses are not alive during all the life of the contract, but are typically written on a small sub-period spanning from 3 to 5 years. At a first glance, this choice may seem strange considering the fact that swing contracts usually have longer maturity, from 10 to 30 years. The main reason why make-up clauses have significantly shorter duration is that no seller takes the risk of giving a buyer the opportunity to move huge quantities of gas for decades. In addition, European gas market illiquidity does not permit realistic forecasts of forward prices. As an example, the longest traded maturity on a very liquid market like the TTF is the 3-year ahead forward, and for longer maturities oil-related instruments are used with much less granularity, so the long term gas term structure is rather flat. In such a situation the decoupling is less marked and the make-up clause loses a bit its importance. For these reasons the market practice is that, if need
arises for the buyer, a new make-up clause can be renegotiated in future years, so the problem of valuing make-up clauses can be split into separate problems.

Let us now concentrate on our 3-year example. Once the make-up quantity of the first year is known, in the second year there are many opportunities: one can call back some (or all) the make-up of the first year or can nominate, if possible, some other make-up that will be called back in the third year.

4.1 Trees quantity

First Year. By Remark 1, we must end the year with

$$0 \leq M_1 = M_1 \leq \min(M, 2M)$$

In fact, in the first year the maximum make-up quantity we can nominate has to be less than or equal to the maximum quantity we can call back in the following two years, that is $2M$, as well as to the maximum quantity we can nominate in a single year. In terms of $z$ and $\tau$, this means that $z = mAQ - \min(M, 2M)$, while $\tau = ACQ$, because we can not call back any previous year make-up, so we have

$$z_{1,12} \in [mAQ - \min(M, 2M), ACQ]$$

Notice that this is in agreement with Equations (21) and (23). Figure 5 shows the possible actions we can perform in the first year.

Second Year. Notice that in this year $M_1 = M_1$: this value strongly influences the possible actions we can take in the second year:

- If $(0 \leq M_1 \leq M)$ we can do one of the following:
  
  i. nominate some other make-up gas in the second year, in such a way that we are able to call back all the make-up in the third year, that is

  $$M_1 + M_2 \leq M \Rightarrow M_2 \leq M - M_1$$

  ii. call back some make-up gas nominated in the first year: the maximum quantity we can call back is, obviously, $M_1$;

  iii. take a quantity of gas between $mAQ$ and $ACQ$, not nominating or calling back any make-up gas.

Summarizing, the constraints for $z_{2,12}$ in this case are

$$z_{2,12} \in [mAQ - (M - M_1), ACQ + M_1]$$

which are again in agreement with Equations (21) and (23). Figure 6(b) shows this case.
Figure 5: Tree quantities for the first year. In the final states where $z_{1,12} < mAQ$, some make-up is nominated and has to be called back in the subsequent 2 years.

- if $M < M_1 (\leq 2\overline{M})$ we must instead call back some make-up gas $U_2$, since otherwise we are not able to arrive at $T_3$ having called back the whole quantity $M_1$. In this case the minimum $U_2$ we have to call back must be such that the final make-up cumulated quantity can be called back in the third year, i.e. $M_2 \leq \overline{M}$, which leads to:

$$M_2 = M_1 - U_2 \leq \overline{M} \Rightarrow U_2 \geq M_1 - \overline{M}$$

Thus, the following constraints for $z_{2,12}$ hold:

$$z_{2,12} \in [ACQ + M_1 - \overline{M}, ACQ + M_1]$$

which now are in agreement with Equations (22) and (23). Figure 6(a) shows this case.

**Third year.** The key quantity now is the residual make-up we have to call back, if any. This quantity, that is exactly $M_2$ as defined in Equation (12), is the residual make-up quantity we must call back in the third year, being this the last year contract (remember we have to call back all the nominated make-up, as seen in Subsection 2.3).
Figure 6: Possible tree quantities for the second year. In subfigure (a), the amount of first-year make-up gas is equal to that which has to be called back in both second and third years: thus, we are forced to end with $z_{2,12} > ACQ$. In subfigure (b), the first-year make-up gas can be called back in a single year, so we have the choice among calling back some quantity ($z_{2,12} > ACQ$), respect the constraints ($z_{2,12} \in [mAQ, ACQ]$) or nominate some other make-up gas ($z_{2,12} < mAQ$).

So there are two cases for this year:

- if $M_2 > 0$ then we have to call back the whole make-up quantity accumulated in the first two years and we have no choice for $z_{3,12}$:
  \[ z_{3,12} = ACQ + M_2 \]
  This case is shown in Figure 7(a)  

- if $M_2 = 0$ then, as we can not nominate any make-up, the constraints for $z_{3,12}$ are given by:
  \[ z_{3,12} \in [mAQ, ACQ] \]
  This case is shown in Figure 7(b)  

Both the cases agree with Equation (18).

5 Sensitivity analysis of a three years contract

A swing contract is a derivative product whose value depends on two main classes of factor, namely market and volumetric. As previously explained in this paper, this kind of derivative shows an optionality value linked to the market price dynamics of the underlying commodity (exercise or not) and an optionality value linked to the volumetric structure of the product itself (how much to allocate with the make-up clause among
the years and how much to withdraw in each subperiod). After having explained how to price a swing product on gas and how to determine the optimal exercise policy, it is now interesting to use the algorithm in order to explore and map the value of the contract with respect to some peculiar parameters of the contract and to market factors.

More in detail, we specify a trinomial dynamics for both the price $P$ and the index $I$ which approximates a geometric mean-reverting Ornstein-Uhlenbeck process as described in Appendix A, and calibrate these models following [8], using historical data on TTF prices for the gas price $P$ and the ENIGR07 formula for the index price $I$. For ease of implementation, the average index price $\Gamma_j$ of year $j$ which appears in Equation (24) is substituted with the average of forward prices for that year. When not variable, the parameters used in this section are the ones in Table 5.

![Diagram](image)

(a) $M_2 = U_3 > 0$

(b) $M_2 = U_3 = 0$

Figure 7: Possible tree quantities for the third year: here the kind of tree totally depends on the cumulated make-up residual quantity $M_2$ from the previous years. If $M_2 > 0$, we are forced to put $U_3 = M_2$ and consume $z_{3,12} = ACQ + U_3 > ACQ$, ending up with a tree as subfigure (a). If $M_2 = 0$, we are forced to satisfy the constraints and consume $z_{3,12} \in [mAQ, ACQ]$, ending up with the tree in subfigure (b).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ACQ$</td>
<td>$7.00 \cdot 10^6$</td>
<td>$\sigma_P$</td>
<td>0.6</td>
</tr>
<tr>
<td>$mAQ$</td>
<td>$6.00 \cdot 10^5$</td>
<td>$a_P$</td>
<td>2.95</td>
</tr>
<tr>
<td>$MDQ$</td>
<td>$8.75 \cdot 10^5$</td>
<td>$\sigma_I$</td>
<td>0.1</td>
</tr>
<tr>
<td>$mDQ$</td>
<td>$3.75 \cdot 10^5$</td>
<td>$a_I$</td>
<td>19.04</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.75</td>
<td>$S$</td>
<td>0</td>
</tr>
<tr>
<td>$r$</td>
<td>0.05</td>
<td>$\rho$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5: Values of the parameters used for the analysis (when not variable).

---


---

25
We here present three analyses: the first one with respect to the volatility level \( \sigma_P \) of gas price, to the \( MDQ \) contract parameter, and to the level of market price decoupling. The second one is done with respect to the level of decoupling of the price term structure and to interest rates level. Finally, the third one is done with respect to correlation between \( P \) and \( I \) and level of decoupling.

The choice of these analyses has been made in view of the goal we are pursuing, namely, to analyse the flexibility given by the make-up clause in a decoupled market scenario. In view of this, we decided to change the parameters we believe to be more impactive on the value of the make-up clause. The volatility \( \sigma_P \) is representative of market uncertainty: in fact, \( \sigma_P \) is often much greater than \( \sigma_I \), since the index \( I \) is calculated as a time average of a basket; as mentioned in the Introduction, this averaging is used to reduce the volatility of the index and leads also to a rather stable value for \( \sigma_I \). Thus, changes in \( \sigma_P \) are likely to influence the price more than those in \( \sigma_I \). The choice of \( MDQ \) is explained by the fact that this quantity is strictly linked with the maximum make-up \( \mathcal{M} \) the owner of the contract can call back in every year. In fact, the bigger \( MDQ \) is, the bigger \( \mathcal{M} \) becomes, and the higher becomes the possibility for the owner to posticipate the calling back of the nominated make-up gas. This flexibility should increase the contract value, in particular when price decoupling is strong. We have decided not to move the minimum quantities. On one hand we set the minimum annual quantity and the minimum period quantity in such a way that the possible make-up one can nominate every year is very high \((1.5 \cdot 10^6)\), so the stronger constraints are on \( \mathcal{M} \). On the other hand, we imposed the values of \( K, \overline{K} \) to be integer and we used values for \( MDQ \) in Table 5. The underlying idea is that any possible increase in the callable make-up quantity \( \mathcal{M} = mAQ - N \cdot mDQ \) is worthless if the upper bound of gas withdrawal per year \( \mathcal{M} \) is not enough to call back the nominated make-up quantity. Thus, we map the contract value for \( MDQ \) in the range between \( \frac{ACQ}{N} = \frac{7.10^6}{12} \simeq 5.83 \cdot 10^5 \), which reduces to the case of a standard contract without make-up clause\(^5\) and a value big enough to ensure the withdrawal in the third year of the possible make-up gas nominated in the first and second year, i.e. bigger than \( \frac{ACQ + 2(mAQ - N \cdot mDQ)}{N} \simeq 8.3 \cdot 10^5 \) and such that \( K, \overline{K} \) are integers.

<table>
<thead>
<tr>
<th>MDQ</th>
<th>( \mathcal{M} )</th>
<th>( \mathcal{M} )</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.83 ( \cdot 10^5 )</td>
<td>0</td>
<td>1.5 ( \cdot 10^6 )</td>
<td>No make up</td>
</tr>
<tr>
<td>6.25 ( \cdot 10^5 )</td>
<td>5 ( \cdot 10^5 )</td>
<td>1.5 ( \cdot 10^6 )</td>
<td>Low flexibility</td>
</tr>
<tr>
<td>8.75 ( \cdot 10^5 )</td>
<td>3.5 ( \cdot 10^6 )</td>
<td>1.5 ( \cdot 10^6 )</td>
<td>High flexibility</td>
</tr>
</tbody>
</table>

Table 6: Values of \( MDQ \) used in the analysis. All the other parameters, when not variable, are set as in Table 5.

The choice of changing \( MDQ \) and not other parameters is also a consequence of currently prevailing practice: we believe that the minimum annual quantity and the

\(^5\)in fact, if \( MDQ = \frac{ACQ}{N} \), then, since it is not possible to call back any make-up gas before having reached \( ACQ \), we are never able to call back any make-up gas, thus it is also impossible to nominate some.
minimum period quantity are less subject to negotiation than the maximum ones: the seller of the contract will never be willing to sell too much flexibility at the expense of his profits (he wants to sell the physical gas), and the buyer will not pay too much for some flexibility he will probably not use in the future (he needs the physical gas).

The second and third analyses focus mainly on market factors. As already stated, the make-up clause becomes profitable for the buyer of the contract only if the spread between market and index price $P_t - I_t$ is expected to be lower in the future than in the present. On the other hand, the make-up gas is paid in two different times and its price is affected by the interest rate, as seen in Eq. (14). Consequently, the benefits of the decoupling could be affected by high levels of interest rates, which could potentially annul the power of make-up clause. This is the focus of the second analysis. Furthermore, the correlation could potentially affect the benefits given by the decoupling: in principle, the decoupling should be enforced by negative correlation and weakened by positive one. This is the subject of the third analysis.

**First Analysis.** The first analysis studies how the contract value depends on the volatility level $\sigma^P$, on the MDQ contract parameter and on the level of decoupling. The latter is obtained by varying the initial forward prices used to calibrate the tree prices (see Appendix A), subtracting a level $S$ from the forward prices $F_t^P$ for the first year and adding the same quantity to the forward prices for the third year, as shown in Subfigure [10(b)]. Then we let $S$ be a parameter and see how the swing price depends on it.

We expect the swing contract value to be increasing in $\sigma^P$, with a higher dependence when there is no flexibility given either by the absence of a make-up clause or by small values of MDQ. Figure 8 shows exactly these qualitative intuitions. The contract value is increasing with respect to $\sigma^P$ also for high values of MDQ, but the range in $y$ axes in the figure is so large that we may not appreciate the monotonicity of the curves. This also evidences the fact that the rights given by make-up reduce the risk given by market uncertainty.

The dependence between contract value and decoupling parameter $S$ is presented in Figure 9. Make-up rights are useful when market decoupling is high. In these situations, we can nominate make-up gas at the beginning of the contract life and call it back in the future, when a positive market scenario shows up.

**Second Analysis.** The second analysis is performed by mapping the swing value with respect to the decoupling parameter $S$ and the interest rate $r$ and reporting the corresponding prices in Figure 11.

The spirit of this analysis is that the make-up clause is exercised when a negative market scenario (typically, contractual price $I$ higher than spot gas price $P$) is expected to change or disappear in the following years through a change in the slope of the index and the gas price forward term structure. On the other hand, as we saw in Subsection 2.3, the make-up gas nominated is paid partly immediately, and partly when the gas is withdrawn; this temporal mismatch implies cash flow effects whose impact obviously
also depend on the level of interest rates: for higher interest rate levels, the benefit of the make-up clause is absorbed by the capitalization of the cost sustained from the end of make-up nomination’s year up to the withdrawal period. Conversely, in a standard contract without make-up clause, a higher interest rate in a market scenario with a low level of decoupling may lead to a higher contract value: in fact, if the decoupling is low, the present value of the contract in the long term, where the swing option is at or out of the money, is lower than the value in the short term, where the option is in the money. Figure 11 shows how any positive change in $S$ is negatively compensated by an increase in the interest rates level.

Third Analysis. The third analysis maps the contract value with respect to the correlation $\rho$ between the two prices $P$ and $I$, and the level of decoupling $S$. In Figure 12(a) we see that decoupling knocks out correlation: in fact, the swing price’s dependence on $S$ is much greater than that on $\rho$, enforcing once again a strong dependence of the swing price on decoupling levels. Only a deeper analysis, performed for fixed values of decoupling, allows a better understanding of the impact of correlation: negative values of $\rho$ leads to higher values of the contract. This is not a surprise: when $\rho$ is negative the decoupling between prices is expected to be stronger (if $P$ rises then $I$ falls because $\rho < 0$) and this increases the value of the contract. However, the changes due to correlation are still smaller than the changes due to decoupling, even for small values of
6 Conclusions

The oil-to-gas price decoupling of recent years, especially since the 2008 financial crisis, has made the make-up clause a very important feature embedded in most long term gas swing deals. In this paper we describe, frame and solve the optimization issue related to the presence of a make-up clause in a swing option. As for a standard swing contract, we show that it is possible to reduce the pricing of a swing option with make-up clause to a stochastic control problem, which can be solved using in a suitable way the Dynamic Programming algorithm. The key idea is to introduce the make-up gas debt as new state variable and incorporating it in the annual constraints on the state space. The dynamic programming is used both on every sub-period of the contract and year by year, by taking into account the gas debt at the beginning of every year. It turns out that, under some not very restrictive assumptions, the optimal withdrawal in all the single sub-periods is of bang-bang type, i.e. it is always optimal to choose in every sub-period between the minimum ($mDQ$) or the maximum ($MDQ$) possible

Figure 9: Sensitivity with respect to the decoupling $S$ and three values of $MDQ$, from no make-up rights to very large flexibility. As expected, decoupling enforces make-up value.
withdrawal quantity. This induces a quantization on the number of distinct sequences of the different optimization problems we have to solve in every year, and this number is shown to be dependent on the range $MDQ - mDQ$ and on the annual upper and lower bounds (ACQ and mAQ, respectively). We prove that the total number of optimization problems to solve is quadratic with respect to the product of the duration $D$ of the contract in years and the number $N$ of considered sub-periods. After having described the full algorithm for a generic number of years $D$, we extend this algorithm to another form of make-up clause, as well as to another clause possibly present in swing contracts, namely the carry-forward.

The algorithm and its extensions are followed by a detailed description of a 3-year contract, which shows how the algorithm works in each year of the contract and how the problem is potentially complex even for small values of $D$. The algorithm of Section 2 is implemented on this 3-years contract, by choosing as the dynamics of $P$ and $I$ a suitable trinomial model with mean-reverting properties, which is calibrated to market data (in particular, TTF for the price $P$ and ENIGR07 for the index $I$) as explained in Appendix A. This implementation is then used to perform a sensitivity analysis of the price with respect to MDQ and some other market parameters, namely the volatility of the spot price $\sigma^P$, the correlation between spot price and market index $\rho$, the interest rates level $r$, and the possible decoupling of gas and index prices induced by the term structure of forward prices, introduced as a perturbation modulated by a parameter $S$ (see Figure 10).

The first conclusion is that the market uncertainty given by volatility can be decreased using make-up rights: high levels of make-up rights lead to a less marked

Figure 10: Scenarios for the term structure of gas and index prices for two levels of decoupling. In subfigure (a) make-up rights are typically not exercised, and prices are not decoupled, while in subfigure (b) typically make-up gas is declared in the first year and called back in the third year thanks to the decoupling.
dependence in contract value compared to a standard contract, where the dependence is more pronounced.

The main conclusion is however that the decoupling induced by forward prices is crucial in assessing whether the make-up clause is a significant component in the price of the swing option. Figure 9 is clear: the higher the value of the make-up rights, the higher the dependence of the price on the decoupling. The slope of the contract value changes completely with high make-up rights, turning the decoupling into a favourable market behaviour. Large values of decoupling $S$ also increase the dependence of the swing price on interest rates: in Figure 12 we show how the make-up is sensible to high values of the risk free rate $r$. The benefits of the decoupling may be in contrast with high rates and make-up, but also in these cases a contract with make-up is always worth more than a contract without. Finally, we investigate the dependence of contract value with respect to correlation $\rho$. It turns out that contract value remains more or less unchanged when correlation changes, so this dependence is much less significant than the dependence on decoupling.

In conclusion, make-up clauses are a powerful tool for managing the new market scenario induced by decoupling and also price uncertainty. We expect that such type of contracts will be traded more frequently in the future, and here we presented a fast algorithm to price them.
Figure 12: Contract value with respect to correlation $\rho$ and level of decoupling. In Figure (a) the shift $S$ vanquishes the effect of the correlation: in fact by varying $\rho$ we obtain almost indistinguishable curves, both with or without make-up. In order to see the differences between curves, in Figure (b) the shift $S$ is fixed; here we can see how correlation affects contract value with make-up: negative values of $\rho$ lead to higher contract values (negative $\rho$ supports decoupling), but the stronger influence of the decoupling $S$ is always evident.

Future work should be aimed at taking proper account of the discrete nature of the index price $I$, as opposed to the continuous evolution of gas price $P$, as well as the implicit non-Markovianity of $I$ which, being a time average over several months, has an evolution with relevant memory effects.

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A Tree Prices

We assume that the log-prices \( X_{j,i} := \log P_{j,i} \) and \( Y_{j,i} := \log I_{j,i} \) follow the discretized version of the mean-reverting dynamics

\[
\begin{align*}
  dX_t &= (\theta_t^P - a^P X_t) \, dt + \sigma^P \, dW_t^P \\
  dY_t &= (\theta_t^I - a^I Y_t) \, dt + \sigma^I \, dW_t^I
\end{align*}
\]

where \( W_t^P \) and \( W_t^I \) are two Brownian motions with mutual correlation \( \rho \): these processes are particular cases of the model in [24] and are rather standard models for energy prices (see for example [14], Chapter 23.3).

In the discretized version, both \( X_{j,i} \) and \( Y_{j,i} \) change at the beginning of every sub-period (i.e. at the beginning of every month). This is exactly what happens for the index \( I \), and it is an acceptable simplification for the gas price \( P \). In particular, we discretize the prices \( (P_{j,i})_{j,i} \) and \( (I_{j,i})_{j,i} \) by building two trinomial trees with the procedure explained in [9, 14] and here summarized.

The first step is to build trinomial trees for \( X \) and \( Y \) by discretizing the dynamics of processes

\[
\begin{align*}
  dX_t^* &= -a X_t^* dt + \sigma dW_t, \quad X_0^* = 0 \tag{33}
\end{align*}
\]

with \( (a, \sigma) = (a^P, \sigma^P) \), or \( (a, \sigma) = (a^I, \sigma^I) \) in the analogous specification for \( Y^* \). The trees for these processes are symmetric around 0 and their nodes are evenly spaced in time and value at intervals of predetermined length \( \Delta t \) and \( \Delta X^* = \sigma \sqrt{3\Delta t} \).

As usual, we denote by \((i, j)\) the node \( x_{i,j} \) in the tree for which \( x_{i,j} = X_t^* \) with \( t = i \Delta t \) and \( X_{i\Delta t}^* = j \Delta X^* \). Hull and White proved [15, 17] that the probabilities to switch from node \((i, j)\) to node \((i + 1, k)\) are always nonnegative if \( -\bar{j} \leq j \leq \bar{j} \), where \( \bar{j} \) is the smallest integer greater than \( 0.184/(a \Delta t) \). This means that at every time step \( i = 0, \ldots, N \) we have a finite number of nodes \((i, j)\) placed at points \( j \Delta X^* \) for every integer \( j \in \{-\bar{j}, \ldots, -1, 0, \ldots, \bar{j}\} \), with \( \bar{j} := \min \{\bar{j}, 2i - 1\} \). Thus, the total width of the tree depends on \( a, \sigma \) and \( \Delta t \).

The second step is to put together the two trinomial trees in a 2-dimensional tree for \((X^*, Y^*)\): this is done at each node in such a way to preserve the marginal distributions of \( X^* \) and \( Y^* \) and the covariance structure induced by the correlated Brownian motions \( W^P \) and \( W^I \), as in [16] (see also [9, Appendix F]).

The third step is aimed to calibrate the previous symmetric tree to the term structure \( F_i \) one has, \( F_i \) standing for the value of the forward with maturity \( i \Delta t \): this step is used to incorporate into the tree mean reversion to levels different from zero, and in particular can be used here to introduce seasonality effects. This is obtained by adding a quantity \( \alpha_i \) to the value \( x_{i,j} \) of all nodes \((i, j)\). For every step \( i \) we have a value for \( \alpha_i \)

\[
\text{Notice that in this Appendix the notation } (i, j) \text{ is not referred to the notation "year } j, \text{ month } i" \text{ used until now in the paper. Here we not distinguish between year and months, having a unique time index } i \text{ that varies between } 0 \text{ and } N \cdot D. \text{ However, for sake of notation, in this appendix we suppose that } i = 0, \ldots, N, \text{ being } N \text{ the appropriate number.}
\]
such that:

\[ \sum_j Q_{i,j} e^{\alpha_i + x_{i,j}} = F_i \]

that leads to

\[ \alpha_i = \log(F_i) - \log \left( \sum_j Q_{i,j} e^{x_{i,j}} \right) \]

having denoting with \( Q_{i,j} \) the probability to reach the node \((i, j)\) starting from the node \((0, 0)\). Once we have the values for \( \alpha_i \) we obtain the final tree which has, at step \( i \), the nodes with value \( e^{\alpha_i + x_{i,j}} \).

An example of two possible final results for the two trees, obtained for some values of \( a \) and \( \sigma \), is plotted in Figure 13. Notice that the higher \( a^f \) (or \( a^p \)) is, the less nodes the respective tree have.

In order to calibrate for the parameters of Equation (33), we use a procedure inspired by [8]. The main idea is to use the discrete time version of the solution of Equation (33):

\[ X^*(t) = X^*(s)e^{-a(t-s)} + \sigma e^{-at} \int_s^t e^{au} dW_u, \quad 0 \leq s < t, \]

which gives

\[ x(t_i) = bx(t_{i-1}) + \delta \varepsilon (t_i) \quad (35) \]

with

\[ b = e^{-a\Delta t}, \quad \delta = \sigma \sqrt{\frac{1 - e^{-2a\Delta t}}{2a}} \quad (36) \]

and \( \varepsilon \) is a Gaussian white noise \((\varepsilon(t) \sim N(0, 1)\) for all \( i \)). Then, in order to provide the maximum likelihood estimator for the parameters \( b \) and \( \delta \), perform a least squares regression of the time series \( x(t_i) \) on its lagged value \( x(t_{i-1}) \), as in Equation (35). Once we have \( b \) and \( \delta \), we can invert Equation (36) and derive the original parameter \( a \) and \( \sigma \).

References


(a) Strong mean reversion, low volatility: 
\[ a^P = 3, a^I = 10, \sigma^P = 0.3, \sigma^I = 0.1 \]

(b) Small mean reversion, high volatility: 
\[ a^P = 0.1, a^I = 0.1, \sigma^P = 0.7, \sigma^I = 0.2 \]

Figure 13: Trees for prices for different values of parameters. Notice that the higher 
\( a^I \) and \( a^I \) are, the less nodes the respective trees have: in subfigure (a) we have trees 
obtained with high \( a^I \), \( a^P \) and few nodes in both the trees, while in subfigure (b) we 
have the converse situation.


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